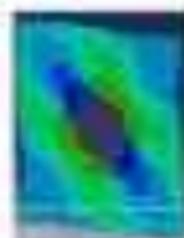
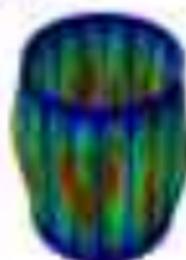


Lecture Notes on Finite Element Methods for Aerospace



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Abstract

This course is intended for those engineers who are interested in the use of finite element methods in aerospace engineering and related disciplines. The course is intended to provide the theoretical background and also include practical aspects of applications. It is designed to assist in understanding design of a lot of primary aerospace structures such as airframes, fins and other structural parts of aircrafts.

With a view that has been stated, the course is being conducted with the objective to help those interested engineers of all related sectors. All rights reserved for the course development and publication of the reference material. This course is available for download by students and engineers who are registered members.

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This work is dedicated to all my teachers who have imparted their knowledge and guidance throughout the course.

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Course Objective

Understand the concepts of mathematical modeling of engineering problems by introducing the Finite Element Methods and to help the students use the method and commercial software packages to solve simple engineering situations.

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Course Objectives

- The student will be able to:
 - Analyze the applications and limitations of FE analysis and its use in the design and optimization of structures and mechanical systems.
 - Apply the finite element method to solve problems involving stress and strain in structures.
 - Use the finite element method to solve problems involving heat transfer and fluid flow in structures.
 - Use the finite element method to solve problems involving dynamic behavior of structures.

Syllabus

Unit 1 Introduction to FE: equilibrium conditions, stress displacement relation, basic constitutive relations – linear elastic, types of elements, assembly procedure, boundary conditions, formulation: Potential energy method, Variational formulation, Rayleigh-Ritz method, Galerkin and Ritz method, Ritz method.

Unit 2 Continuum mechanics, compatible strains, 1D structures: axial elements from formulation, derivation of shape functions, problems using 1D elements, Beam formulation: direct formulation and derivation of shape functions and problems – 2D structures: Plane stress and Plane strain elements formulation, shape function development, problems using 2D elements – 2D structures: elements requirements, formulation of elements.

Unit 3 3D element formulation: Introduction to 3D formulation of Plane truss and Shell elements – Variational approach – Solution techniques of the structural systems: Application to FE systems: FE modeling of structural and spatially continuous systems – of boundary conditions and loading on FE models, Analysis of substructures by using linkage, matrix condensation.

Overview

Dr. J. J. Egan: "The Environmental Protection Agency and the 1970s"

Overview

The 1970s: "The Environmental Protection Agency and the 1970s"

Dr. J. J. Egan: "The Environmental Protection Agency and the 1970s"

Robert J. Egan: "The Environmental Protection Agency and the 1970s"

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Introduction

Why the Course Fits

The field of Mechanics can be subdivided into 3 disciplines:



Computational Mechanics

Branch of Computational Mechanics can be distinguished according to the physical level of analysis



4 Computational Solid and Structural Mechanics

A systematic introduction of problems in Computational Solid and Structural Mechanics (CSM)

4 components
 (Solid and Structural
 Mechanics I & II)

Statics

Dynamics

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CSM Statics

A further introduction of problems in CSM Statics

CSM Statics

Statics

Dynamics

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CSEI Linear Methods

For the analysis of conditions on the computer we sometimes choose a special alternative method:



The **finite difference method (FDM)** is a numerical technique that solves differential equations by approximating the differential coefficients with difference quotients.

The **boundary element method (BEM)** is a numerical technique used to solve linear partial differential equations. It is based on discretization and is used to solve and approximate a wide problem in elasticity.

The **finite volume method (FVM)** is a numerical method for approximating PDEs by dividing the domain into cells and evaluating the integral of the flux over each cell.

The **finite element method (FEM)** is a numerical technique for solving partial differential equations that consists of the discretization of the domain with the addition of special elements.

CSSE Linear-Statics by FEM

Having solved the FEM for displacement, we can solve any problem formulated and solved as follows:

Formulation of FEM Model

Displacement
Function
Mass
Stiffness

Solution of FEM Model

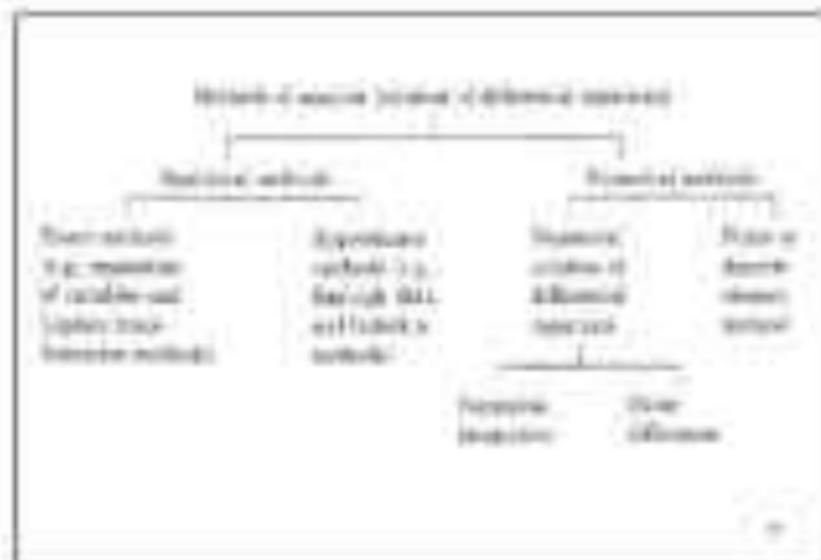
Stiffness
Function
Mass

Field of Mechanics :

- Statics
- Solid
- Continuum
- Gas/Fluid

Computational mechanics is energy formulation





Theoretical Mechanics

Deal with fundamental laws and principles of mechanics starting from first principles (such as conservation laws)

Formal Mechanics

Translates the theoretical knowledge to technical and engineering applications by construction of mathematical models of physical phenomena

Computational Mechanics

Implements specific models by computer numerical methods with numerical methods implemented in digital computers (FEM, FEM, etc.)

Experimental Mechanics

Real physical tests, mathematical models and numerical simulation are validated when compared with experimental results

Most of the Solid Mechanics Problems are Boundary Value Problems (BVP):

Any solution to this problem must satisfy the following conditions.

Field Condition: Field variables, displacements (strain), stresses (stress resultants) must satisfy the governing equations (GE).

GE - Differential Equations (DE) or Variational Formulations (VF) and

Boundary Condition (BC) - Kinematic and static.

Differential Equations:

Governing equations (GE) derived by variational treatment of the principles of mechanics.

GE is also obtained using Euler-Lagrange equations.

DE for beam (Euler-Bernoulli):

$$EI \left(\frac{d^4 w}{dx^4} \right) = P$$

DE for plate (Kirchhoff):

$$\left(\frac{d^4 w}{dx^4} \right) + 2 \left(\frac{d^4 w}{dx^2 dy^2} \right) + \left(\frac{d^4 w}{dy^4} \right) = \frac{p}{D}$$

where,

$$D = \frac{EN^3}{12(1 - \nu^2)}$$

(i) What is the Finite Element Method?

The Finite Element Method (FEM) is a sophisticated numerical scheme to approximate the approximate solution to various boundary-value problems of engineering and mathematical physics.

- The finite element method is a numerical method for solving problems of engineering and mathematical physics.
- Useful for problems with complicated geometries, loadings, and material properties where analytical solutions can not be obtained.

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The Finite element method is a numerical method for solving problems of engineering and mathematical physics.

Engineering Problems:

- Structural analysis
- Heat transfer
- Fluid flow
- Mass transport and
- Electromagnetic potential

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- ▶ Algebraic and analytical mathematical solutions are obtained via matrix operations (geometry, boundary and internal properties used in FEM method).
- ▶ Analytical solutions are those given by a mathematical expression that states the values of the desired unknown quantities of one location in a body and are thus valid for an entire volume of material in the body.
- ▶ Analytical solutions typically involve solving ordinary or partial differential equations (convenient to complex geometries, varying load conditions, and diverse material properties). These solutions are often unavailable (there is no need to rely on numerical methods, such as finite element method for acceptable solutions).

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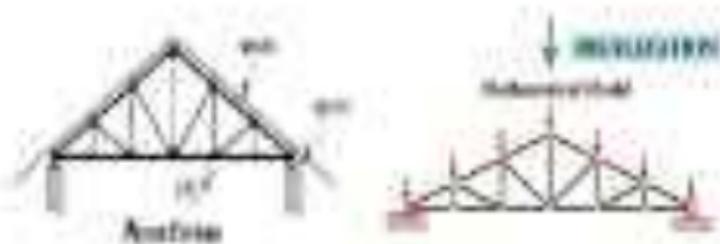
Discretization:

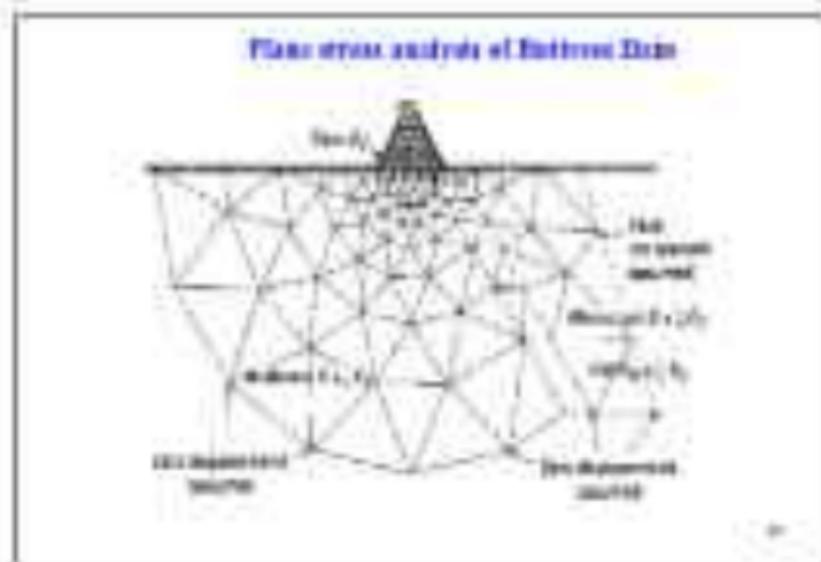
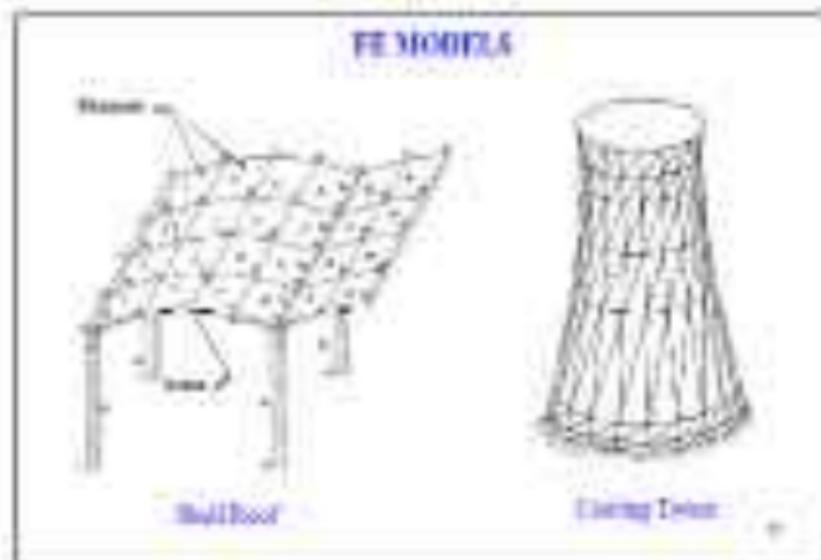
modeling a body by dividing it into an equivalent system of smaller bodies (or) discretized at points common to two or more elements (nodes) and/or boundary lines and/or surfaces.

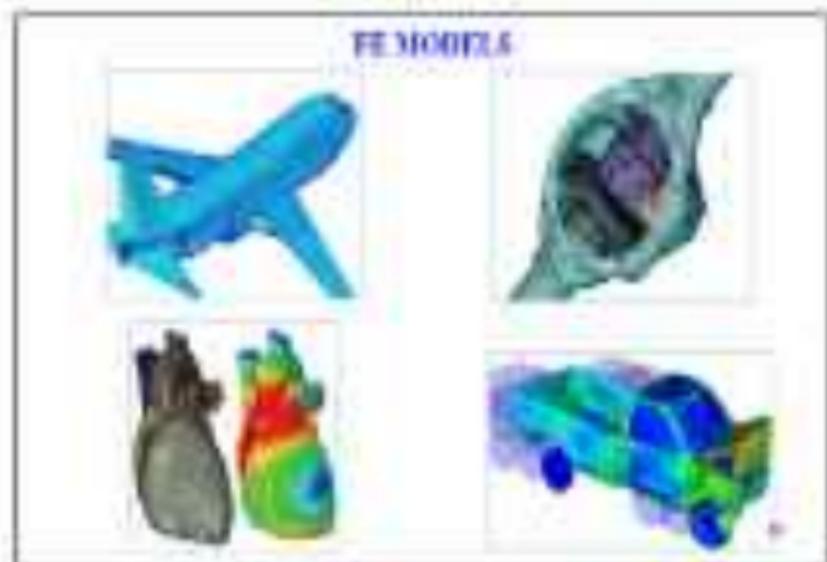
The solution for structural problems typically refers to determining the displacement (deflection) of each node and the stresses (stress) which arises due to loading on the structure that is subjected to applied loads.

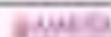


- Model the body by dividing it into an equivalent system of smaller bodies (finite elements) discretized at points common to two or more elements (nodes) and/or boundary lines and/or surfaces.









Non-Structural Problems: The initial unknown may be temperature or fluid pressures due to thermal or fluid fluxes.

General Steps of FEM:

Two general approaches associated with the FEM:

- 1) Force or Flexibility Method
- 2) Displacement or Stiffness Method

Flexibility method



It is used to obtain stress as the unknown solution. To obtain the governing equations, first the equilibrium equations are used. Then, the necessary additional equations are found by introducing compatibility equations. The result is a set of algebraic equations for determining the unknowns and unknown forces.

Stiffness Method

Assumes the displacement of the nodes as the unknowns of the problem. Compatibility condition requires that elements assembled at a common node, along common edge or on a common surface before loading remain connected at the node. Independent shape deformation functions are initially assumed.

Then, the element matrices are obtained in terms of nodal displacements using the equations of equilibrium and as applicable, by relating known displacements.

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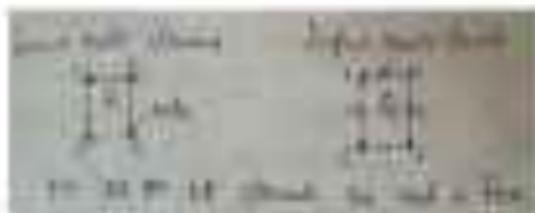
- Dimensional property, the displacement method is more suitable because of its flexibility to apply to nearly all kinds of problems and used as the general purpose (or) universal and basic design method.
- The main involves totalizing the structure using small interconnected elements called finite element a displacement function is associated with each finite elements.
- Every interconnected element is linked, directly or indirectly to some other element through common interfaces, including nodes, boundaries, boundary conditions.
- Using unknowns like displacement, stresses, strains, etc. the behavior of finite parts in terms of the properties of finite element element in their solutions.
- The total set of equations describing the behavior of finite parts results in a series of algebraic equations, here expressed in matrix notation.

FINITE ELEMENT METHOD

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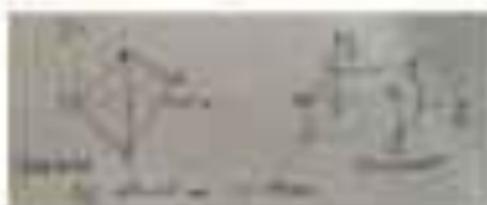
I. Discretize and select the element types

1. breaks down the body into a specified number of finite elements.
2. the finite element types are engineering judgement.
3. optimal element is used to reduce computational effort.



@AMBA

- The slide between 14, 20, 21 or otherwise even shows method of M analysis as a critical tool for designers, analyzing the concept of stress and strain applications for accuracy and efficiency of the simulation.



@AMBA

- Defining a point loading and generally should be asymmetric.
- stress (σ) triangular stress through 3d (mg)



FINITE ELEMENT METHOD

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2. Introduction of the displacement function

- Involves choosing a displacement for every node element
- The function is defined at the element using the nodal values of the element
- Linear, quadratic, and cubic polynomials are frequently used.
- For example, an element of length L is used for a 2D element the displacement function is a function of displacement in x plane.

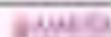
**3. Define the term shape function and its role**

- Describes displacement and its derivatives over the element for deriving the equations for each finite element. $\psi_i(x)$ is the shape function for node i where:

$$\psi_i = \psi_i(x) \quad \text{for each element}$$

- It is possible to find the shape function for a through a 1-D bar using Castiglino's law.
- The ability to define the shape function is necessary to avoid the need to obtain the shape function via the

$$\psi_i = \psi_i(x)$$



Choosing the element stiffness matrix and equations:

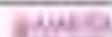
➤ Direct equilibrium method:

- If there is no load and external equations (e.g. boundary forces), no nodal displacements are calculated using force equilibrium conditions for each element, along with force/deformation relationships.
- This method is extremely necessary for 2D elements (e.g. quadrilaterals, triangles), and beam elements are essential components in structural analysis.



➤ Work energy method:

- This method is valid to 2D and 3D elements for developing (K) for 2D and 3D elements.
- Principles of virtual work (virtual displacement), the principle of minimum potential energy and Biot's theorem (strain energy) are used to derive the element stiffness matrix.
- Principles of virtual work applicable for any material behavior, principle of minimum potential energy and Castiglione's theorem are applicable only to elastic materials.
- Functional methods analogous to be used with the principle of minimum potential energy and also work is virtual (K) matrix and influence the heat transfer, fluid flow problems.



Let $\Pi = \Pi(x, y, z)$

- $\Pi(x, y)$ denote a function f of two variables x and y .
- We have $\Pi = \Pi(x, y, z)$
- Where Π is a function of the location T

$$\Delta \Pi = \iint_{\Omega} |\nabla \Pi|^2 |\mathbf{J}| |\mathbf{J}|^T d\mathbf{r}$$

Elementary volume



Method of Weighted residuals (WR)

- Useful for developing the standard standard computer visualization method
- This method used for case studies as the exact method whenever the wrong method are available
- The WR are generally applicable & facilitated to it as particular energy based methods possible
- The scheme that elements can be divided to various element through any of the methods

$$\int \mathbf{W} = \mathbf{W} \cdot \mathbf{W}$$



Where \mathbf{W} = Weight of element nodes \mathbf{W} error

\mathbf{W} = General element matrix nodes (all nodes of element)

\mathbf{W} = Weight of nodes \mathbf{W} displacement, etc. see below.

FINITE ELEMENT METHOD @ANMISA

5. Assemble the element equations to obtain the global equations and introduce boundary conditions as follows:

- The internal nodes equations matrix added together using a method of superposition allows method to get global equations equation

$$[F] = [K] [U]$$

$[F]$ - Global nodal force vector

$[K]$ - Global stiffness

$[U]$ - unknowns

$[F]$ - global matrix because it determines a results are

- To minimize computer storage, we need to do it carefully so that the unknown nodes is kept as small as possible as a result.

FINITE ELEMENT METHOD @ANMISA

6. Solve for the unknown Dof:



- set of unknowns and stiffness equations in a matrix of unknown code/ dof to take very alternative method/ give method of a- to solve method (given table) if answer quality

6.1.10.1

 7. Take for the element e and σ :

 • In e certain are used to define σ

$$C_e = \frac{\partial \sigma}{\partial \epsilon} \rightarrow A_{\text{ref}} = \frac{\partial \sigma}{\partial \epsilon} + \frac{\partial \sigma}{\partial \epsilon} \rightarrow$$

8. Interpret the results:

- Study element and analyze the results to see if the design/analysis process described the area concerned, manufacturing and other aspects etc.
- Post processors compute some other items to present the results.

6.1.10.2



various steps involved in the finite element analysis are:

- (i) Select suitable field variables and the elements.
- (ii) Describe the domain.
- (iii) Select appropriate functions.
- (iv) Find the element properties.
- (v) Assemble element properties to get global properties.
- (vi) Impose the boundary conditions.
- (vii) Solve the system equations to get the nodal unknowns.
- (viii) Make the additional calculations to get the required values.

$$[K] \{U\}_n = \{F\}_n$$

$$[K] \{U\} = \{F\}$$

where $[K]$ is stiffness matrix

$\{U\}$ is displacement vector and

$\{F\}$ is force vector in the coordinate directions

Finite Element Method

- Discretization (Spatialization, approximation, and computer realization)
- FEM is a semi-discrete spatial approximation method
- Underpinned by CAE applications
- Applications
 - Structural Engineering
 - Mechanical Engineering (Stress, Temperature)
 - Thermal Anal.
 - Acoustics
 - Hydraulics
 - Electromagnetics
 - ...

Advantages

- Irregular Geometries
- Dynamic Loads
- Different Materials
- Boundary Conditions
- Variable Element Size
- Easy Identification
- Nonlinear Problems (Geometric, Material, Contact, Buckling)

In FEM, all three of load, material and stress load, both loads and responses of stress and responses, are discretized in space time with a finite order.

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Objectives of This FEM Course

- Understand the fundamental ideas of the FEM
- Know the behavior and usage of each type of element around it this course
- Be able to prepare a suitable FE model for given problems
- Can interpret and evaluate the quality of the results (know the physics of the problems)
- Be aware of the limitations of the FEM (also know the FEM - a numerical tool)

Computer Implementations

- Preprocessing (build FE model, loads and constraints)
- FEA solver (assemble and solve the system of equations)
- Postprocessing (sort and display the results)

Available Commercial FEM Software Packages

- ABAQUS (General purpose FE and non-linear)
- ANSYS (General purpose FE and FEA package)
- PATRAN (General purpose FE pre-processor)
- ALGOR (Nonlinear and dynamic analysis)
- COSMOS (General purpose FE)
- ADINA (FE and non-linear)
- ZIBLAD (FE, FEA, Dynamics)
- STRESS (FE and FEA)
- IDEFES (Dynamic FE analysis)

Advantages of General Purpose Programs

- + Easy input - preprocessor.
- + Solves many types of problems
- + Modular design - Fluid, dynamics, heat, etc.
- + Can run on PC's now.
- + Relatively low cost.

Disadvantages of General Purpose Programs

- High development costs.
- Less efficient than special programs.
- Often proprietary, their source is code locked.



Feature

- Obtain a set of algebraic equations to solve for unknown (total quantity (displacement)).
- Secondary quantities (stresses and strains) are expressed in terms of nodal values of primary quantity.



Steps of Derivation of the Finite Element Method:
1. Derivation of the element:
1.1 Derivation of the cross-sectional geometry of displacement function:

$$\begin{aligned}
 \text{displacement } u &= 0 \\
 \text{displacement } v &= 1 \\
 \text{displacement } w &= 0
 \end{aligned}$$

1.2 Derivation of element stiffness matrix and load vector:

$$k = \int \sigma \epsilon dx$$

1.3 Assembly of global stiffness matrix to obtain the global coefficient matrix:

$$U = \sum u_i$$

1.4 Solution for the unknown global displacement, $\{K\}U = \{F\}$
1.5 Calculation of element forces and stresses:

$$\text{Stress } \sigma = E \epsilon \quad \text{Stress } \sigma = E \frac{du}{dx}$$

Types of Finite Elements
1D (Line) Element

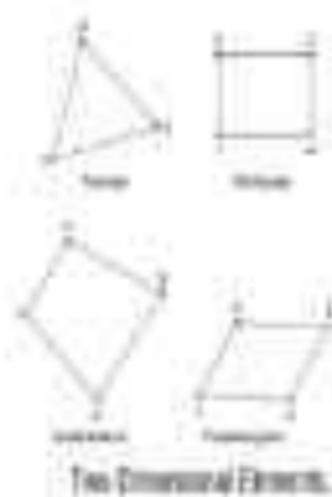
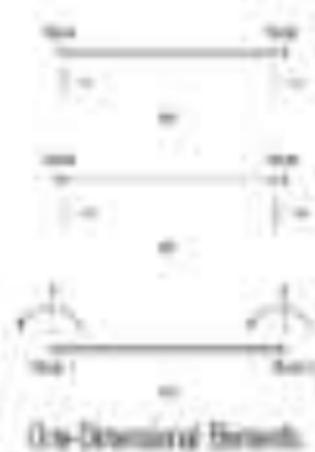

(spring, bar, beam, pipe, etc.)

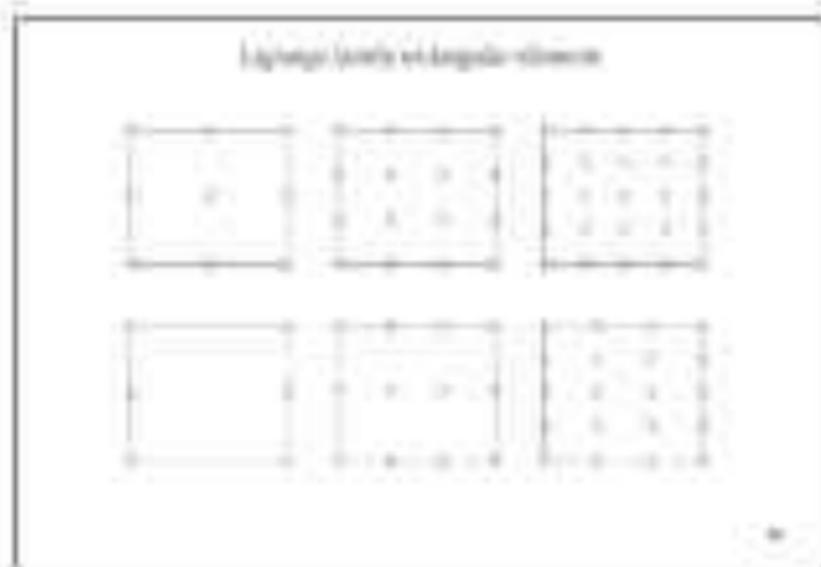
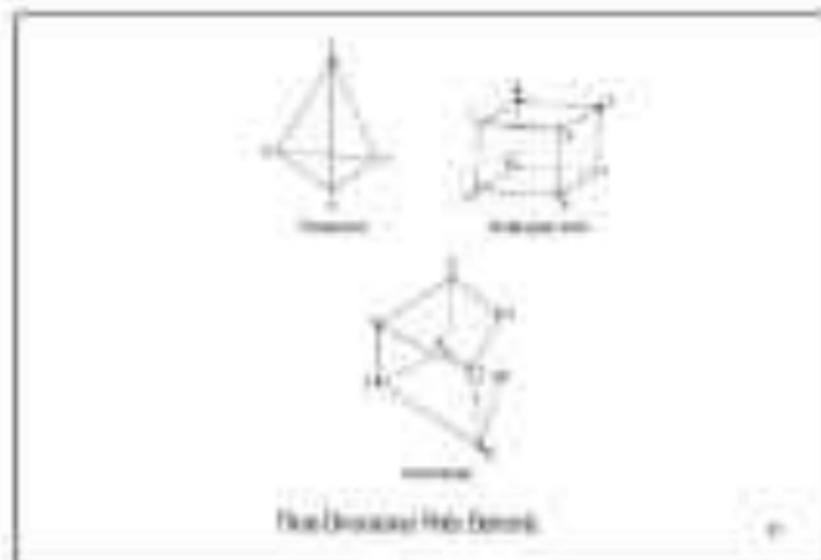
2D (Plane) Element


(triangle, plate, shell, etc.)

3D (Solid) Element


(brick, pipe, structural member, etc.)

BASIC ELEMENT SHAPES


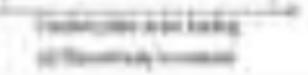
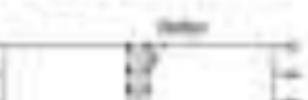
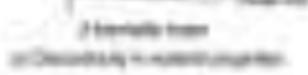
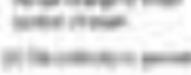
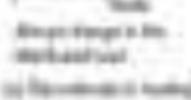
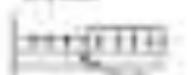


Simply supported rectangular domain



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Location of Nodes



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Number of Elements

The number of elements in an analysis is determined by extent to the required desired level of accuracy and the volume of data involved. Although the volume for the number of elements generally increases with increase in size, it is not given priority since all the same volume of three-dimensional solids is a way, about is significantly important. This behavior is shown graphically in figure 1.10. Increased rates through of a large number of elements results in large volume of data, we have to be able to store the results resulting in the computer resources required.



FIG. 1.10.10. Effect of Data

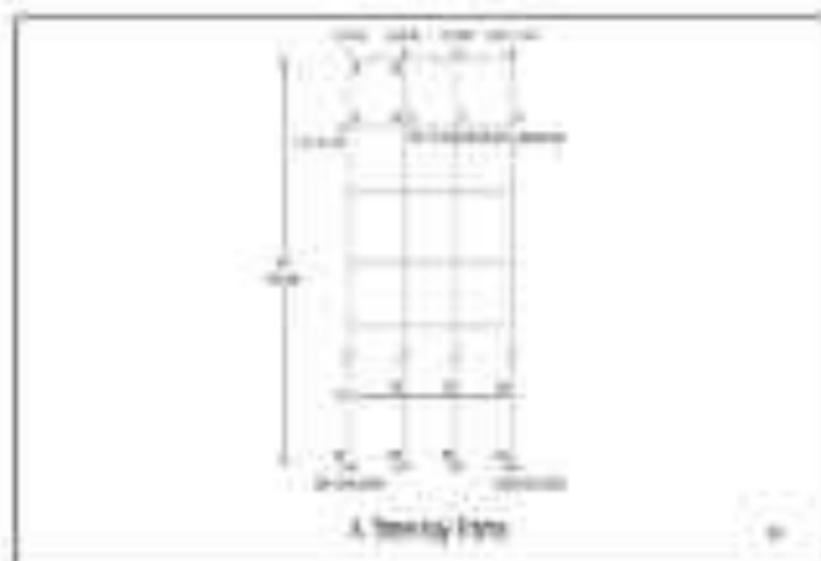
NODE NUMBERING SCHEME

• The III analysis involves matrix equations in which the unknown variables will be totaled.

• Large III problems provide due to limited volume of the resources.

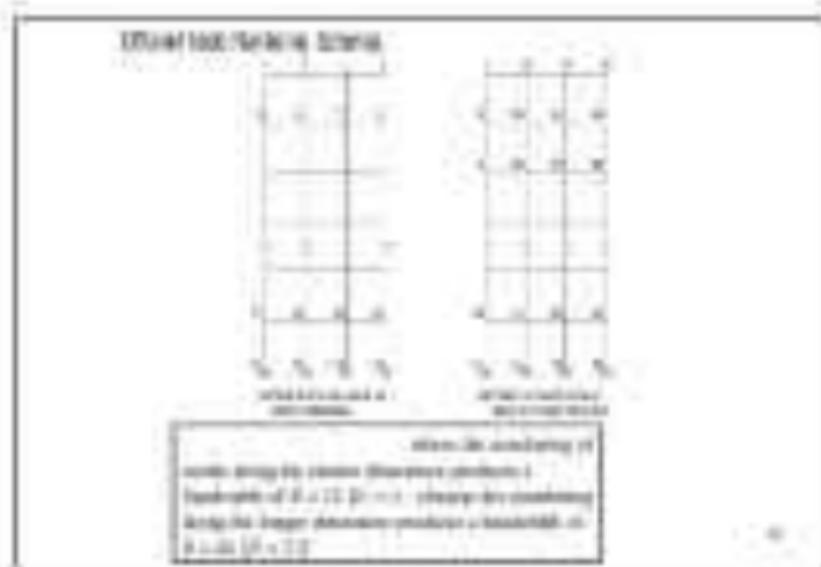
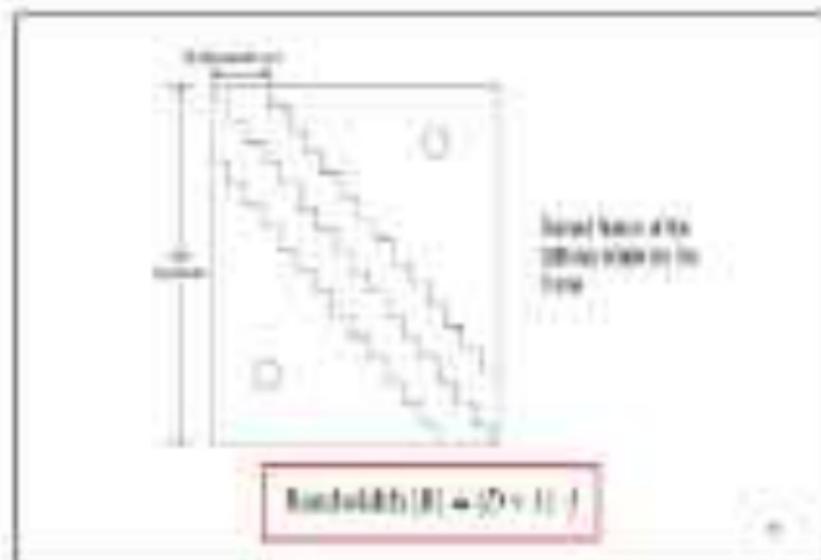
• Most of analysis are non-linear, complex systems can be substantially reduced by storing only the elements involved in full band width instead of storing the entire matrix.

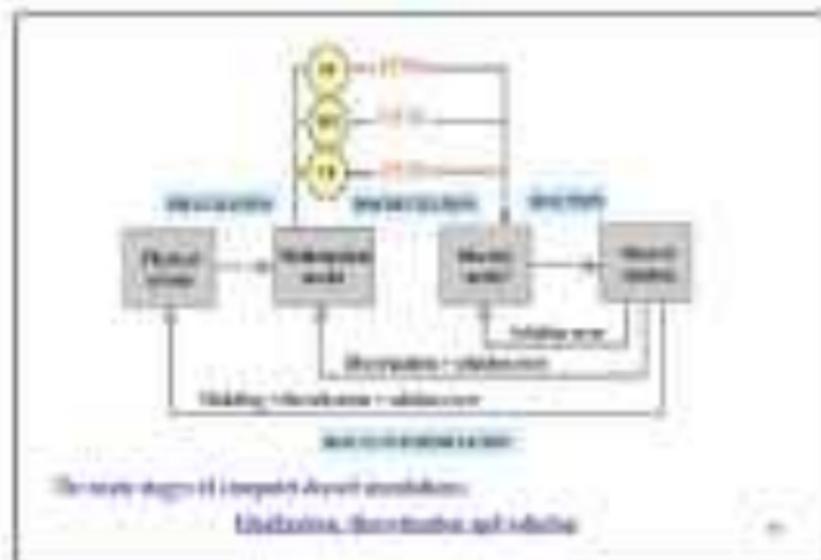
• The bandwidth of the global stiffness matrix depends on the node numbering scheme and the number of dof considered per node.



As an exercise, consider
 a three-story frame with rigid joints, 20 stories high.

Assuming that there are 4 dof per node, there are 252 unknowns in the final equations (including the dof corresponding to the fixed nodes), and if the entire stiffness matrix is stored in the computer, it will require $252^2 = 63,504$ locations. The bandwidth (strictly speaking, half bandwidth) of the overall stiffness matrix can be shown to be 43, and thus the storage required for the upper half-band is only $43 \times 252 = 10,836$ locations.





Strong Form (SF)

Treated as a system of ordinary (or partial differential) equations to solve with the usual computational approaches (finite difference, finite element, etc.). As long as they reduce to algebraic equations.

Weak Form (WF)

Treated as a weighted integral equation that allows the usage of hierarchical classes of shape functions.

Functional Form (FF)

Treated as a functional minimization problem; involves gradient descent and other optimization.

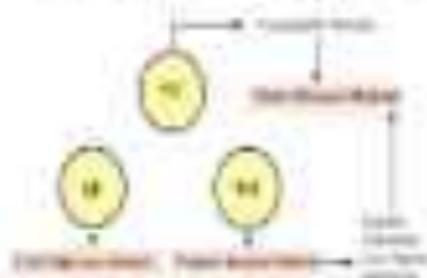
Requires a discrete set of nodes and is dependent on the discretization used for the finite element.

Why Weak and Variational Forms?

The following reasons may be offered:

1. Basis for important direct methods of approximation, notably FEM
2. Characterizes essential quantities of interest to engineer
3. Checks treatment of boundary & interface conditions
4. Provides useful mathematical treatment of questions of existence, stability, error, etc. Also provides guidelines to help to achieve desirable behavior in finite models

Mathematical Model Forms as Sources of Discretization Methods



Strong, Weak and Variational Forms as sources of numerical approximation methods

Derivation of Element Matrices and Vectors

The element matrices and vectors of finite elements can be derived by using any of the following approaches:

I. Direct Approach

In this method, direct physical reasoning is used to substitute the element properties in terms of potential variables.
 Applicability of several types of elements

II. Variational Approach

The FE analysis is interpreted as an approximation method for solving variational problems.
 Most physical field engineering problems can be formulated in variational form. The FEM can be readily applied for finding their approximate solutions.
 Major limitation – requires the physical field variational problem to be stated in variational form, which may not be possible in all cases.

I. General Variational Approach

The element matrices and vectors are derived directly from the governing differential equations.
 Can be applied to almost all the practical problems of stress and displacement. Efficient Procedures: Galerkin method, Least square method.

- If the physical problem is given in a variational problem, and the exact solution, which satisfies the integral, are not the given exactly, we would like to find an approximate solution that approximately satisfies the integral.
- Instead of the variational problem in terms of differential equations and B.C., and/or the variational solution, which satisfies all the conditions from B.C., can not be obtained exactly, we would like to find an approximate solution that satisfies the integral exactly by using governing differential equations.

Variational Formulation

Essence: minimize value of the functional Π sought.

The governing function for SO: total potential energy or total complementary energy.

Using the calculus of variation, stationary value of a functional obtained by equating first variation of the functional to zero.

$$\delta \Pi = 0$$

The solution to the field variables, when (after stationary value is by Π is obtained) $\delta \Pi$, also should satisfy all the prescribed boundary BC's.

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Boundary Conditions

1. Essential BC (displacement, slope, derivatives of slope) and
2. Natural or Stress BC (force, moment)

Typical deflection w or any point value as field variable, it must be satisfy the BC, which is an equilibrium condition, or $\frac{\delta \Pi}{\delta w} = 0$.



Example 10.1.1

W (slope) is fixed (clamped)

Free or Stress BC at B

$$\text{shear} = \frac{\delta \Pi}{\delta V} = 0$$

$$\text{M} = \frac{\delta \Pi}{\delta M} = 0$$

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Potential Energy

Potential energy is the capacity of the forces to do work. For structural systems or deformable bodies the forces are classified as external forces due to loads and internal forces due to stresses (or stress resultants). Thus the total potential energy is the sum of the potential energy of the external and internal forces.



The free spring shown in Fig. 1, as the reference is, the spring changes from P to x , the strain energy of the spring is $\frac{1}{2}Px^2$. Thus, the total potential energy of the spring is

$$U = \frac{1}{2}kx^2 - Fx \quad (1.6)$$

The differential w (elemental work) done due to increase in displacement is

$$dW = \sigma_x \phi \, dx \left(1 + \frac{\partial \phi}{\partial x} dx\right) \left(\frac{1}{2} \sigma_x \, dx\right) \\ = \sigma_x \phi \left(\frac{\partial \phi}{\partial x}\right) dx \, dx \quad (2.8)$$

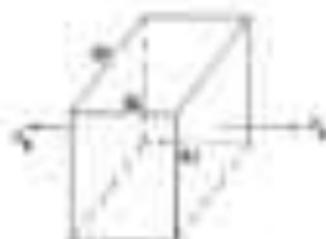


Fig. 2.1 The element differential work

Since, $\frac{\partial \phi}{\partial x} = \epsilon_x$, we get,

$$dW = \sigma_x \phi_x \, dx \, dx \quad (2.9)$$

As the strain becomes large due to the load value ϵ_x , the differential work done which is the strain energy developed in the element

$$\int_V \sigma_x \phi_x \, dV$$

If the element is subjected to all the six stress components, the total strain energy developed in the element is given by

$$\int_V \sigma_x \epsilon_x + \int_V \sigma_y \epsilon_y + \int_V \sigma_z \epsilon_z \\ + \int_V \tau_{xy} \gamma_{xy} + \int_V \tau_{yz} \gamma_{yz} + \\ \int_V \tau_{zx} \gamma_{zx} \, dV$$

This expression is preferred form useful in the final state of stress.

Let U_0 be the strain energy density, i.e., strain energy per unit volume. Then, generalizing Eq. 3.18 we can express the total strain energy density as

$$U_0 = \sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx} \quad (3.19)$$

From differential calculus we know that U_0 can be expressed as

$$U_0 = \frac{\partial U_0}{\partial \epsilon_x} \epsilon_x + \frac{\partial U_0}{\partial \epsilon_y} \epsilon_y + \frac{\partial U_0}{\partial \epsilon_z} \epsilon_z + \\ \frac{\partial U_0}{\partial \gamma_{xy}} \gamma_{xy} + \frac{\partial U_0}{\partial \gamma_{yz}} \gamma_{yz} + \frac{\partial U_0}{\partial \gamma_{zx}} \gamma_{zx} \quad (3.20)$$

Comparing Eqs. 2.11 and 2.5, we get

$$\frac{\partial U_1}{\partial v_1} = e_1, \quad \frac{\partial U_1}{\partial v_2} = e_2, \quad \frac{\partial U_1}{\partial v_3} = e_3 \quad (2.50)$$

Thus, the units being directly for the property that the partial derivatives with regard to any finite component is the corresponding force component.

Eq.2.50 can be expressed in matrix form as,

$$\left\{ \frac{\partial U_1}{\partial v} \right\} = \{e\} \quad (2.51)$$

Substituting Eq. 2.14 from Eq. 2.11 into Eq.2.5, we get

$$\left\{ \frac{\partial U_1}{\partial v} \right\} = [C] \{v\} \quad (2.52)$$

Integration of Eq.2.52 with respect to the static fields

$$\begin{aligned} U_1 &= \int_V (v^T C^T) dV \\ &= U_1 = \frac{1}{2} v^T P \{v\} \end{aligned} \quad (2.53)$$

In the case of linear elastic isotropic body, substituting for C^T from Eq.2.14, assuming $V = V_0$ and simplifying we get,

$$\begin{aligned} U_1 &= \frac{1}{2E} (e_1^2 + e_2^2 + e_3^2) - \frac{\nu}{2E} (e_1 e_2 + e_2 e_3 + e_3 e_1) \\ &= \frac{E + \nu}{2} (e_1^2 + e_2^2 + e_3^2) \end{aligned}$$

(9)

$$U_e = \frac{1}{2} (k_1 u_1^2 + k_2 u_2^2 + k_3 u_3^2 + k_4 u_4^2 + k_5 u_5^2 + k_6 u_6^2)$$

The total strain energy, U , can be obtained by integrating U_e over the volume of the solid. Thus,

$$U = \iiint_V U_e dV \quad (2.26)$$

And substituting for U_e from Eq.2.21,

$$U = \frac{1}{2} \iiint_V \sigma^T \epsilon dV$$

4

In case initial stress and strain, and temperature effects are to be considered, the stress-strain relation given by Eq. 2.11 is to be used for the evaluation of strain energy. It can be shown,

$$\{\sigma\} = [D] \{\epsilon\} - \{\sigma_0\} + \{\sigma_1\} \quad (2.18)$$

$$U = \iiint_V \frac{1}{2} \{\sigma\}^T \{\epsilon\} dV = \frac{1}{2} \{\epsilon\}^T [D] \{\epsilon\} dV$$

In the above equation, a term, $\frac{1}{2} \{\epsilon\}^T [D] \{\sigma_0\} dV$ is constant and has no effect in the variational process in order of equilibrium equation.

4

Then the total potential energy Π is defined as the sum of the internal potential energy and strain energy

$$\Pi = \mathcal{U} + \mathcal{E}$$

Substituting from Eq. 2.47 and 2.6b,

$$\Pi = \frac{1}{2} \iiint_V (\sigma^T \epsilon) dV - \iint_V (w^T \gamma) dV - \iint_{\Gamma} w^T \bar{t} dS$$

Substituting for ϵ and γ $\epsilon = \mathcal{D}u$ $\gamma = \mathcal{D}u|_{\Gamma}$

$$\Pi = \frac{1}{2} \iiint_V w^T \mathcal{D}^T \sigma \mathcal{D} u dV - \iint_V w^T \gamma dV - \iint_{\Gamma} w^T \bar{t} dS$$

where σ is the stress of the element and γ is the portion of the boundary where tractions are applied.

According to the principle of minimum potential energy, the first variation of Π must be zero for equilibrium condition. Thus taking the first variation we get

$$\delta \Pi = \delta U^e - \delta W^e = \int_{\Omega} \delta \sigma^T \epsilon \, d\Omega - \int_{\Omega} \delta u^T \rho b \, d\Omega - \int_{\Omega} \delta u^T \rho^T \alpha \, d\Omega = 0$$

$$\int_{\Omega} \delta u^T \rho b \, d\Omega = \int_{\Omega} \delta \sigma^T \epsilon \, d\Omega - \int_{\Omega} \delta u^T \rho^T \alpha \, d\Omega = 0$$

$$\mathbf{K} \mathbf{U} = \mathbf{Q} \mathbf{1}$$

In case the element is subjected to thermal loads and stresses, the strain energy U^e ,

$$\int_{\Omega} \int \delta u^T \rho b \, d\Omega - \int_{\Omega} \int \delta \sigma^T \epsilon \, d\Omega = \int_{\Omega} \int \delta u^T \rho b \, d\Omega - \int_{\Omega} \int \delta u^T \rho^T \alpha \, d\Omega = 0$$

Then the equilibrium equation is reduced to the vector form,

$$\mathbf{K} \mathbf{U} = \mathbf{Q} \mathbf{1}$$

where

$$\mathbf{K} = \int_{\Omega} \int \delta \sigma^T \epsilon \, d\Omega$$

$$\mathbf{Q} = \int_{\Omega} \int \delta u^T \rho b \, d\Omega + \int_{\Omega} \int \delta u^T \rho^T \alpha \, d\Omega$$

Formulation of Finite Element Equations (Static analysis)

Two elements of identical material are fixed up to each other, as shown.

Find DOF to be discretized as follows:

Step 1: The stiffness matrix of each element

Step 2: The assembled contribution of element 1 is assembled



$$k = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} = kK^e \quad \text{D.O.F}$$

Here K^e is the element stiffness matrix, k is the stiffness of the element, and D.O.F is the number of D.O.F's.

4

Step 3: The element stiffness matrices are assembled into the global stiffness matrix. The global stiffness matrix is formed if the total is a matrix of the assembly of the two elements.

$$K = \sum_{e=1}^n k$$

Here K^e is the element stiffness matrix and n can be 1000.

$$K = \left(\int \int \int \sigma \epsilon \right) + \left(\int \int \int \sigma \epsilon \right) + \left(\int \int \int \sigma \epsilon \right) \quad \text{etc}$$

Here K^e is the element stiffness, K^e is the global stiffness if the element are distributed within the structure. If we assemble each of the element stiffness matrix.

5

The next step is forming the stiffness matrix K for the element. This is done by integrating the strain energy over the element volume.

$$\begin{aligned}
 U &= \frac{1}{2} \int_V \epsilon^T \sigma \, dV \\
 &= \frac{1}{2} \int_V \mathbf{B}^T \mathbf{u} \mathbf{B} \mathbf{D} \mathbf{u} \, dV \\
 &= \frac{1}{2} \mathbf{u}^T \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \mathbf{u} \\
 &= \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{K} &= \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \\
 &= \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \\
 &= \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV
 \end{aligned}$$

The stiffness matrix \mathbf{K} is obtained by integrating the strain energy over the element volume.

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \quad (14.1)$$

The stiffness matrix \mathbf{K} is obtained by integrating the strain energy over the element volume.

Integration of Eq. (2.77) over Ω (see Fig. 2.78) yields the overall energy of the element e :

$$\begin{aligned}
 e &= \int_{\Omega} \left(\frac{1}{2} \sigma^T \epsilon + \sigma \epsilon - \frac{1}{2} \sigma^T \sigma \right) d\Omega \\
 &= \int_{\Omega} \sigma^T \epsilon d\Omega - \int_{\Omega} \frac{1}{2} \sigma^T \sigma d\Omega
 \end{aligned} \quad (2.78)$$

In the FE (2.76) and (2.78) and in both we either have the constant stress σ or the constant strain ϵ over the whole element. Thus we can factor σ or ϵ out of the integrals. Since the value of each stress (strain) is the same at the two integration nodes ξ of the element (see Fig. 2.78), the total element energy of the element e can be expressed as

$$e = \frac{1}{2} \sigma^T \epsilon V_e \quad (2.79)$$

where $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ is the vector of nodal displacements of the element e (see Fig. 2.78) and V_e is the total volume of the element e (see Fig. 2.78).

Integration of the last two terms in equation (2.79) yields

$$\begin{aligned}
 e &= V_e \left[\frac{1}{2} \left(\frac{1}{2} \sigma^T \epsilon + \sigma \epsilon \right) \right] = V_e \left[\frac{1}{2} \left(\frac{1}{2} \sigma^T \epsilon + \sigma \epsilon \right) \right] \\
 &= V_e \left[\frac{1}{2} \left(\frac{1}{2} \sigma^T \epsilon + \sigma \epsilon \right) \right] = V_e \left[\frac{1}{2} \left(\frac{1}{2} \sigma^T \epsilon + \sigma \epsilon \right) \right]
 \end{aligned} \quad (2.80)$$

Equation (2.80) expresses the total element energy of the element e in terms of the stress σ and strain ϵ . The last two terms in equation (2.80) are identical to using the following common notation for the summation of nodal energy:

$$\frac{1}{2} \sigma^T \epsilon = \frac{1}{2} \sigma_1 \epsilon_1 + \frac{1}{2} \sigma_2 \epsilon_2 + \dots + \frac{1}{2} \sigma_n \epsilon_n \quad (2.81)$$

The element by element form can be written as

$$\underbrace{\left(\sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \nabla u_h \cdot \nabla v_h \right)}_{\text{local bilinear form}} = \underbrace{\int_{\Omega} f v_h}_{\text{local linear form}}$$

and a global bilinear form

\mathcal{B}_h	$\sum_{e \in \mathcal{T}_h} \left(\int_{\Omega_e} \nabla u_h \cdot \nabla v_h \right)$	$\int_{\Omega} f v_h$	$\int_{\Omega} \nabla u_h \cdot \nabla v_h$
local or	local bilinear	local linear	local bilinear
global	global bilinear	global linear	global bilinear
form	form	form	form
	\mathcal{B}_h	\mathcal{L}_h	\mathcal{B}_h
	local bilinear and linear, \mathcal{B}_h		

 and a global linear form \mathcal{L}_h

 and a global bilinear form \mathcal{B}_h

$$\int_{\Omega} \nabla u_h \cdot \nabla v_h = \sum_{e \in \mathcal{T}_h} \int_{\Omega_e} \nabla u_h \cdot \nabla v_h \quad (1)$$

then

$$\mathcal{B}_h = \int_{\Omega} \nabla u_h \cdot \nabla v_h \quad \text{local bilinear form} \quad (2)$$

$$\mathcal{L}_h = \int_{\Omega} f v_h \quad \text{local linear form} \quad (3)$$

$$\mathcal{B}_h = \int_{\Omega} \nabla u_h \cdot \nabla v_h \quad \text{global bilinear form} \quad (4)$$

$$\mathcal{L}_h = \int_{\Omega} f v_h \quad \text{global linear form} \quad (5)$$

Step 1: The element nodal values are used to determine the element shape functions (see Eq. 14.11) as

$$N_1 = 1 - \xi \quad (14.12)$$

and

$$N_2 = \xi \quad (14.13)$$

and

$$N_3 = \frac{1}{2}(1 - \xi) \quad (14.14)$$

Step 2: The element shape functions are used to determine the element shape functions (see Eq. 14.11) as

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State of Stress at a point

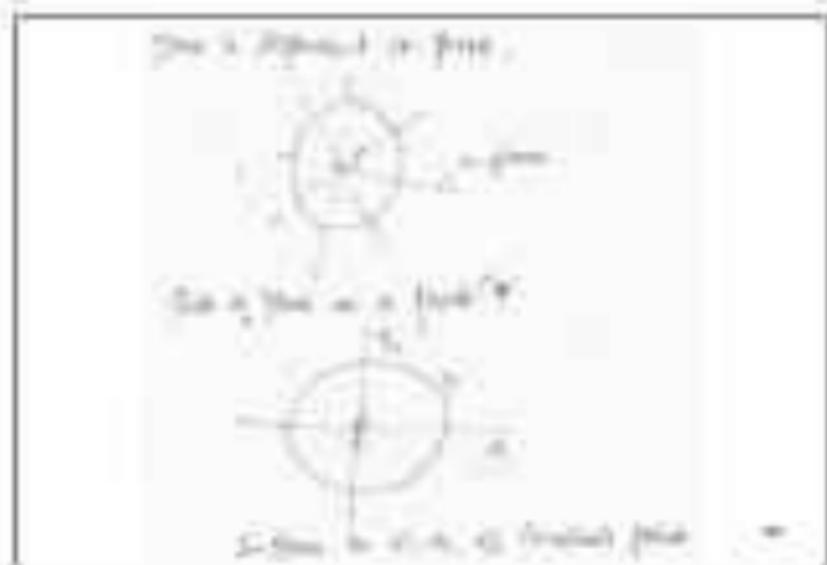
The handwritten notes describe the state of stress at a point. The diagrams show a 3D coordinate system with x, y, and z axes. A point is shown with a small cube element. The stress components are labeled as normal stress (σ) and shear stress (τ). The notes describe the stress components and their relationship to the stress tensor.

$\sigma_x, \sigma_y, \sigma_z$ (Normal stress)
 $\tau_{xy}, \tau_{yx}, \tau_{yz}, \tau_{zy}, \tau_{zx}, \tau_{xz}$ (Shear stress)
 The stress tensor is represented as a 3x3 matrix:

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The stress components are related to the stress tensor as follows:

- σ_x is the normal stress in the x-direction.
- σ_y is the normal stress in the y-direction.
- σ_z is the normal stress in the z-direction.
- τ_{xy} is the shear stress in the xy-plane.
- τ_{yz} is the shear stress in the yz-plane.
- τ_{zx} is the shear stress in the xz-plane.







$$[K] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[F] = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$[K]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[U] = [K]^{-1} [F]$$

$$[U] = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[F] = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$[K]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[U] = [K]^{-1} [F]$$

$$[U] = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

Linear Constitutive Equations

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Linear Constitutive Equations

The constitutive law expresses the relationship among stresses and strains. In theory of elasticity, usually it is considered as linear.

one-dimensional stress analysis, the linear constitutive law is stress is proportional to

strain and the constant of proportionality is called Young's modulus.

It is very well known as Hooke's law: stress is proportional to strain and the constant of proportionality is called Young's modulus.

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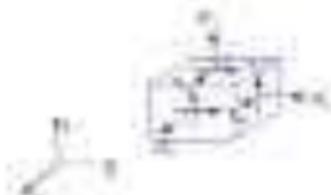
Finite Element for 3-D Problems

In general, the stresses and strains in a continuum element of an isotropic solid

$$\sigma_{ij} = \sigma_{ji}, \quad \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{11} = \sigma_{22} = \sigma_{33} \quad \text{No. of stress}$$

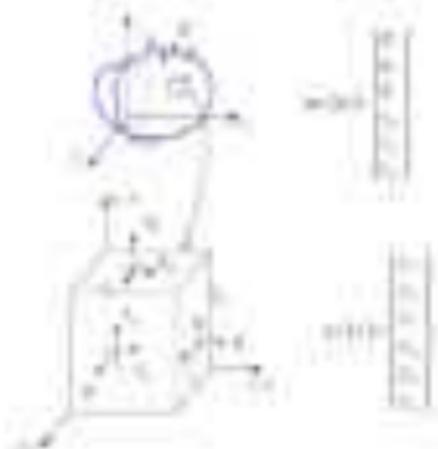
and

$$\tau_{ij} = \tau_{ji}, \quad \tau_{12} = \tau_{21}, \quad \tau_{13} = \tau_{31}, \quad \tau_{23} = \tau_{32} \quad \text{No. of stress}$$



Under volume constraint, the state of stress and strain can be represented in general, in 3-D by the following relations for isotropic elastic continuum.

Stress-Strain for 3-D Problems



Stress-strain relation:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \frac{E}{1+\nu} \begin{pmatrix} \epsilon_{11} + \nu \epsilon_{22} + \nu \epsilon_{33} \\ \nu \epsilon_{11} + \epsilon_{22} + \nu \epsilon_{33} \\ \nu \epsilon_{11} + \nu \epsilon_{22} + \epsilon_{33} \\ \frac{1-\nu}{2} \epsilon_{12} \\ \frac{1-\nu}{2} \epsilon_{13} \\ \frac{1-\nu}{2} \epsilon_{23} \end{pmatrix} + \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{pmatrix} \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \\ \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \\ \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Systems Equations:

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 &= 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 &= 0 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= 0 \end{aligned}$$

Isotropic:

An **isotropic** body has material properties that are the same in every direction at a point in the body. I.e., the properties are independent of orientation at a point in the body.

Orthotropic:

An **orthotropic** body has material properties that are different in three mutually perpendicular directions at a point in the body and, further, has three mutually perpendicular planes of material property symmetry. Thus, the properties depend on orientation at a point in the body.

Anisotropic:

An **anisotropic** body has material properties that are different in all directions at a point in the body. No planes of material property symmetry exist. Again, the properties depend on orientation at a point in the body.

The nodal unknowns are expressed using the six components of stresses and strains and is called **Generalized Hooke's Law**. This may be stated as:

$$\begin{Bmatrix} u \\ v \\ w \\ \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$$\{u\} = [D] \{\sigma\}$$

*

where D is a 6×6 matrix of constants of elasticity to be determined by experimental investigations for each material. As D is symmetric matrix [16 - 23], there are 21 material properties for **isotropic Anisotropic Materials**.

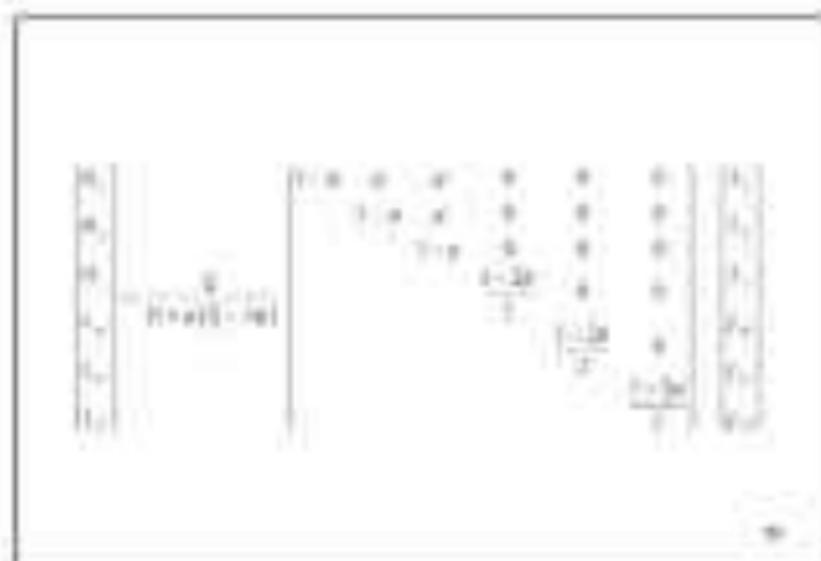
*

It is an ordered sextet symmetry with respect to planes within the body. Such materials are called **orthotropic** materials. Hence for orthotropic materials, the number of constants reduce to 9 as shown below

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 \\ & & D_{33} & 0 & 0 & 0 \\ & D_{21} & & D_{31} & 0 & 0 \\ & & & & D_{23} & 0 \\ & & & & & D_{32} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}$$

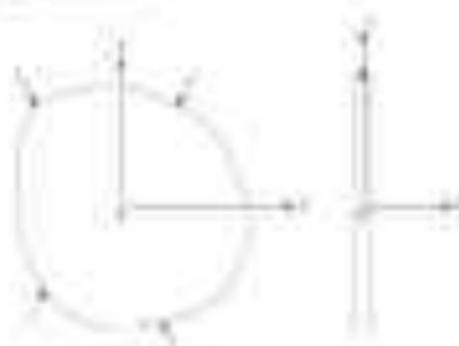
strain stress relation for isotropic material is

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \begin{pmatrix} \frac{1}{E} & \frac{\nu}{E} & \frac{\nu}{E} & 0 & 0 & 0 \\ \frac{\nu}{E} & \frac{1}{E} & \frac{\nu}{E} & 0 & 0 & 0 \\ \frac{\nu}{E} & \frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}$$



Plane Stress Problems

The thin plates subject to loads in their plane only, fall under this category of the problems.



we have to find stress and strain along the length of a beam. Thus

$$\sigma_x = \tau_{xy} = \tau_{yx} = 0$$

The condition $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$ and other condition is, as given

$$\sigma_x = \mu \epsilon_x + \mu \epsilon_y + (1 - \mu) \gamma_{xy} = 0$$

$$\epsilon_x = -\frac{\mu}{1 - \mu} (\epsilon_y + \gamma_{xy})$$

→

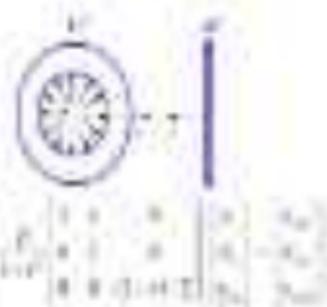
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - \mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 - \mu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

→

a) Shear stress

$$\sigma = \tau_x, \tau_y = 0, \tau_z = 0$$

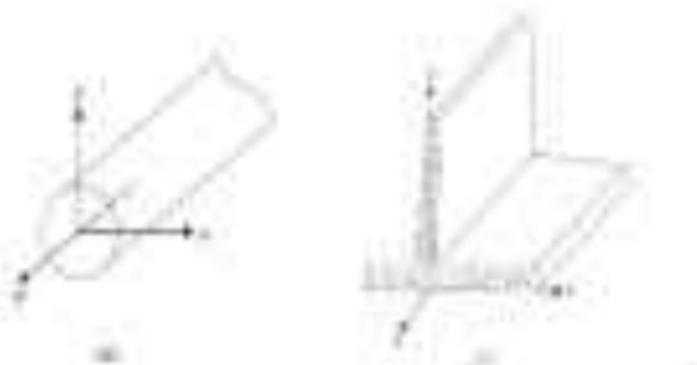
b) Shear stress varies with distance from axis
 (due to shear flow in the circular section)



or

Plane Strain Problems

A long body subject to significant lateral forces for very thick longitudinal joints is to be analyzed in terms of plane strain.



$$\epsilon_x = \gamma_{xz} = \gamma_{zx} = 0$$

$\gamma_{yz} = \gamma_{zy} = 0$ means τ_{yz} and τ_{zy} are zero.

$$\epsilon_y = 0 \text{ means}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu \frac{(\sigma_x + \sigma_z)}{E} = 0$$

$$\sigma_y = \mu(\sigma_x + \sigma_z)$$

4

$$\begin{pmatrix} u_x \\ \sigma_x \\ \tau_{xz} \end{pmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{pmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xz} \end{pmatrix}$$

5

• Plane stress

$$\sigma_x, \sigma_y, \tau_{xy} \neq 0 \quad \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

It may be used with a thickness in the vertical and horizontal directions along the length direction.



$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E/(1-\nu^2) & \nu E/(1-\nu^2) & 0 \\ \nu E/(1-\nu^2) & E/(1-\nu^2) & 0 \\ 0 & 0 & E(1-\nu)/2(1+\nu) \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

in the plane stress case.

Axis-Symmetric Problems

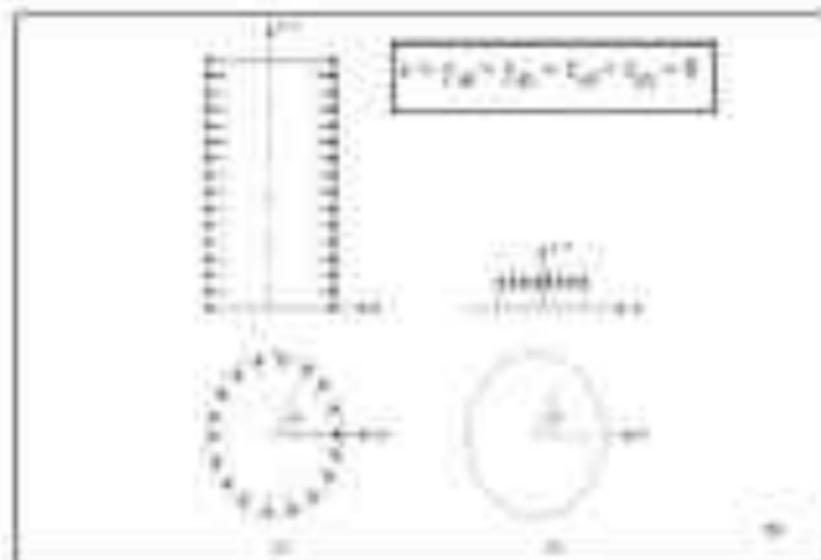
Axis-symmetric structures are those which can be generated by rotating a flat or curved shape around a cylinder.

are the common examples of axis-symmetric structures. If such structures are subjected to

axis-symmetric loading, the analysis is reduced to a 2D problem, with a half model of the total structure.

over the surface.





Here, there are only four unknown components. The exact displacement (table 6) for these components are:

$$u_1 = \frac{qL^4}{24}, \quad v_1 = \frac{qL^4}{8}, \quad \theta_1 = \frac{qL^3}{6} \text{ and}$$

$$u_2 = \frac{qL^4}{24} - \frac{qL^4}{8}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \end{bmatrix} = \frac{q}{(1 + \nu)(E I_x + I_y)} \begin{bmatrix} 1 + \nu & \nu & 0 & 0 \\ \nu & 1 + \nu & 0 & 0 \\ 0 & 0 & 1 + \nu & 0 \\ 0 & 0 & 0 & 1 + \nu \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \end{bmatrix}$$

Stiffness Method

Stiffness Method

k_{ij} - local case
 K_{ij} - global case



k_{11}, k_{22} - local case free
 k_{12}, k_{21} - local case dependent (not)



4. Region
 This method is used to find the value of δ at a point in the structure.
 The δ is the displacement at that point.
 A. Displacement of a point in a structure
 $\delta = \sum \frac{P \cdot \Delta}{AE}$ (Equation 1)
 Add the δ at each point in the structure.
 $\delta = \sum \left\{ \frac{P \cdot \Delta}{AE} \right\}$ (Equation 2)

The δ at a point is given by
 $\delta = \sum \frac{P \cdot \Delta}{AE}$ (Equation 1)
 In a structure $\delta = \delta_1 + \delta_2 + \dots + \delta_n$ (Equation 2)
 $\delta = \delta_1 + \delta_2$
 $\delta = \frac{P_1 \cdot \Delta_1}{AE} + \frac{P_2 \cdot \Delta_2}{AE}$ (Equation 3)

For eqn. (1) and (2)

$$Q_1 = \frac{d_1}{L} \left(\frac{d_2 - d_1}{L} \right) \frac{L}{2} \\ \text{Similarly } Q_2 = \frac{d_2}{L} \left(\frac{d_2 - d_1}{L} \right) \frac{L}{2} \left(\frac{d_2 + d_1}{2} \right) \frac{L}{2}$$

$$Q_1 = \left[1 - \frac{d_1}{d_2} \right] \left\{ \frac{d_1}{2} \right\} \frac{L}{2}$$

$$Q_2 = \left[1 - \frac{d_1}{d_2} \right] \left\{ \frac{d_2}{2} \right\} \frac{L}{2}$$

$$\text{Total } Q = \frac{d_1}{2} \left(1 - \frac{d_1}{d_2} \right) \frac{L}{2}$$

As per Fig. - the shear diagram

By applying eqn. (1) and (2) we get



and $\{d\}$ is vector

$$\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{bmatrix} N & -N \\ -N & N \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$\{F\} = [K]\{d\}$ - system eqn

$$\{F\} = \begin{bmatrix} N & -N \\ -N & N \end{bmatrix}$$

↑
system matrix

② Find the global stiffness equation to obtain global nodal and element DOF

$$K = [K]_1 + [K]_2$$

$$K = \begin{bmatrix} N & -N \\ -N & N \end{bmatrix} + \begin{bmatrix} N & -N \\ -N & N \end{bmatrix}$$

$$\begin{aligned}
 & \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 dx = \frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 dx \\
 & \text{So we need} \\
 & \text{to find } u \text{ such that } \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 dx = \frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 dx \\
 & \text{for } u = \frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 dx \\
 & \text{Then } \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
 & \text{Then } \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
 \end{aligned}$$

Elastic force is not linearly dependent of some
 but a linear one displacement.

This is called continuity in compatibility
 requirement.

$$\mathbf{A}_n^{(1)} = \mathbf{A}_n^{(2)} = \mathbf{A}_n$$



$$F = k \cdot \Delta$$

— global nodal displacement

$$k = \begin{bmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{bmatrix} \quad \text{global stiffness matrix}$$

Assembly of the global matrix by Superposition

For two elements e_1 and e_2

$$K_1 = \begin{bmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{bmatrix} \quad K_2 = \begin{bmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{bmatrix}$$

Then $K = \begin{bmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{bmatrix} + \begin{bmatrix} k & 0 & -k \\ 0 & k & -k \\ -k & -k & 2k \end{bmatrix} = \begin{bmatrix} 2k & 0 & -2k \\ 0 & 2k & -2k \\ -2k & -2k & 4k \end{bmatrix}$

$$k \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad \text{--- (1)}$$

In equation (1) we take

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad \text{--- (2)}$$

Use eq (2) in (1)

$$k \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \Rightarrow \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

Apply eq (2)

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

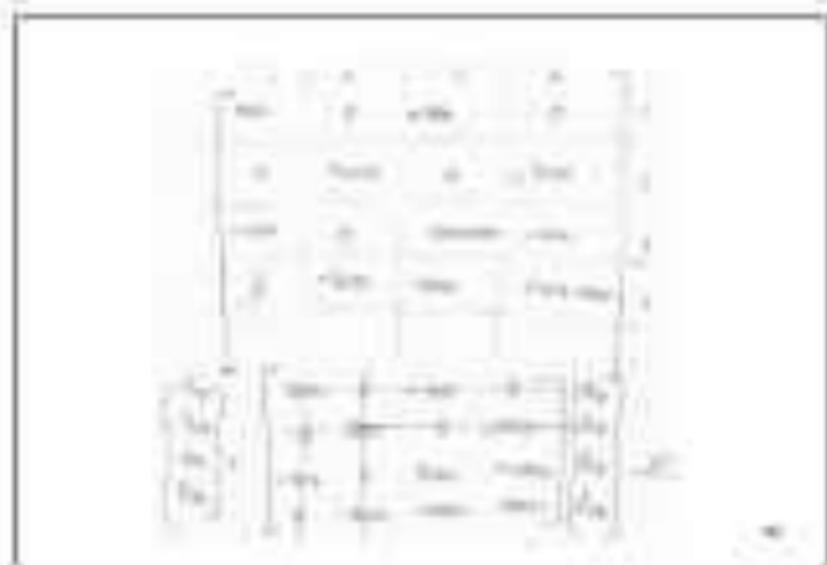
$$[F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$[U] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$[K] \{U\} = \{F\}$$

Properties of [K] matrix

- It is symmetric
- It is singular and has no inverse unless all supports for the system are used to completely fix point and body motion
- The main diagonal terms are always positive values. $\sum_{i=1}^n k_{ii} = \text{rank}(K)$ (rank of matrix)



Be dia = 0 dia = 0 and apply boundary

$$\begin{Bmatrix} 0 \\ 1200 \end{Bmatrix} = \begin{bmatrix} 5000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} \delta_{10} \\ \delta_{20} \end{Bmatrix}$$

$$\delta_{10} = 0.41 \text{ mm}$$

$$\delta_{20} = 1.166 \text{ mm}$$

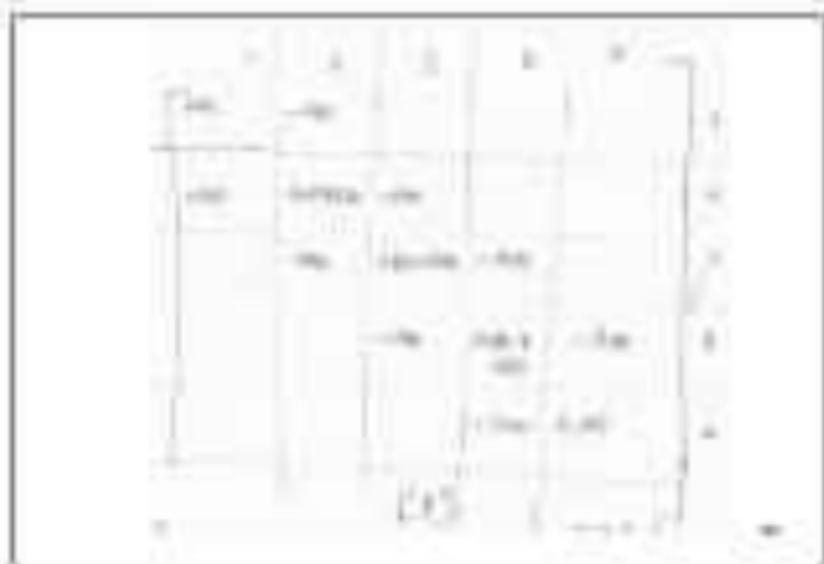
$$\begin{matrix} \text{Element} & \text{Node} & \text{DOF} & \text{Value} \\ \hline 1 & 1 & 1 & 0 \\ & 2 & 1 & 0.41 \\ & 2 & 2 & 1.166 \\ \hline 2 & 2 & 1 & 0.41 \\ & 3 & 1 & 0 \\ & 3 & 2 & 0 \end{matrix}$$

Final value
 $\delta_{10} = 0.41$
 $\delta_{20} = 1.166$
 $\delta_{30} = 0$
 $\delta_{40} = 0$

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega} \rho \mathbf{u} \, d\Omega &= \int_{\Omega} \rho \frac{d\mathbf{u}}{dt} \, d\Omega \\
 &= \int_{\Omega} \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \, d\Omega \\
 &= \int_{\Omega} \rho \frac{\partial \mathbf{u}}{\partial t} \, d\Omega + \int_{\Omega} \rho \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega \\
 &= \int_{\Omega} \rho \frac{\partial \mathbf{u}}{\partial t} \, d\Omega + \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dA \\
 &= \int_{\Omega} \rho \frac{\partial \mathbf{u}}{\partial t} \, d\Omega + \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dA
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega} \rho \mathbf{u} \, d\Omega &= \int_{\Omega} \rho \frac{d\mathbf{u}}{dt} \, d\Omega \\
 &= \int_{\Omega} \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \, d\Omega \\
 &= \int_{\Omega} \rho \frac{\partial \mathbf{u}}{\partial t} \, d\Omega + \int_{\Omega} \rho \mathbf{u} \cdot \nabla \mathbf{u} \, d\Omega \\
 &= \int_{\Omega} \rho \frac{\partial \mathbf{u}}{\partial t} \, d\Omega + \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dA \\
 &= \int_{\Omega} \rho \frac{\partial \mathbf{u}}{\partial t} \, d\Omega + \int_{\partial \Omega} \rho \mathbf{u} \cdot \mathbf{n} \, dA
 \end{aligned}$$

1. The figure shows the finite element mesh for a beam of length L and cross-sectional area A . The beam is fixed at the left end and free at the right end. The beam is divided into four elements of equal length $L/4$. The nodes are numbered 1, 2, 3, 4, 5 from left to right. The displacement at node 5 is denoted by u_5 .

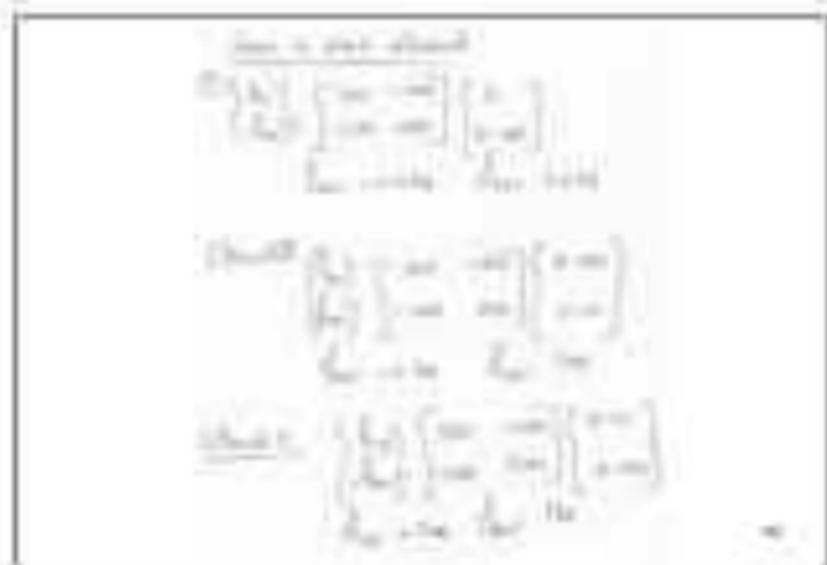


$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$\frac{1}{2} \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$\frac{1}{2} \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$
 $\frac{1}{2} \frac{1}{2} = \frac{1}{4}$



Example 11

$$\begin{Bmatrix} d_{11} \\ d_{12} \end{Bmatrix} = \begin{bmatrix} 200 & -100 \\ -100 & 200 \end{bmatrix} \begin{Bmatrix} 0.10 \\ 0.05 \end{Bmatrix}$$

$$d_{11} = 10 \quad d_{12} = 15$$


 All the $d_{11} = 0$, $d_{12} = 0$, $d_{21} = 0$

Compatibility Condition at node

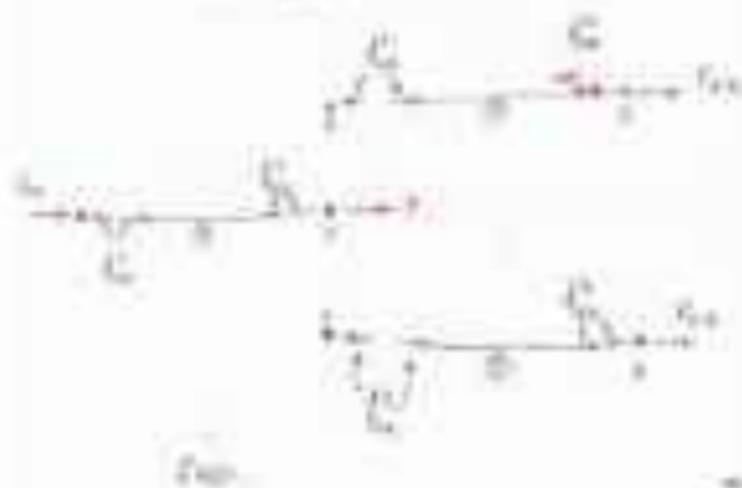
$$d_{11} = d_{21} \text{ and } d_{12} = d_{22}$$

Compatibility Condition at node

$$d_{12}^e = d_{21}^e = d_{12}^e = d_{21}^e$$

→ nodal displacement same

$$\left. \begin{array}{l} k_1 d_1 \\ k_2 d_2 \\ k_1 d_1 + k_2 d_2 \\ k_1 d_1 \\ k_2 d_2 \end{array} \right\} \rightarrow \text{Node}$$





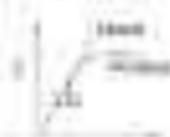
Spring Element



Nodes:	2
Node 1 location:	$x_1 = 0$ m
Node 2 location:	$x_2 = 10$ mm
Spring constant:	$k = 10^6$ N/m

Spring force-displacement relationship:

$$F = k(u_2 - u_1)$$



Consider an equilibrium element for a spring element:

$$F_1 + F_2 - F_3 = 0 \quad (1)$$

where,

$$F_1 = F_2 = k(u_2 - u_1) \quad (2)$$

is given by:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

or,

$$[k] \{u\}$$

Now, derive Spring Element:

- 1. Node 1 location
- 2. Node 2 location
- 3. The value of force-displacement

4. Spring constant

5. Force-displacement curve

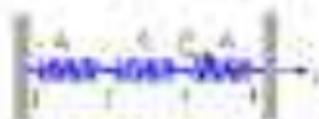
6. Element stiffness matrix

Properties of stiffness matrix

- Invert of stiffness matrix corresponds to flexibility
- Stiffness matrix is symmetric and positive definite
- The number of surface nodes is a multiple of 2 of each face (used to prevent self-interaction)
- Symmetric stiffness matrix means force is directly proportional to displacement
- Diagonal terms of the matrix are always positive i.e. force/displacement is always positive
- Largest term will be seen in diagonal entry of the structure's stiffness



Example



Given: For the spring system we have

$$k_1 = 100 \text{ N/m}, k_2 = 200 \text{ N/m}, k_3 = 100 \text{ N/m}$$

$$F = 100 \text{ N}, u_1 = u_3 = 0$$

Find: (a) the global stiffness matrix

$$(b) \text{ the displacement of node 2 (mm)}$$

$$(c) \text{ the reaction force at node 1 (kN)}$$

$$(d) \text{ the force in the spring 2}$$

(c) **Stiffness matrix**

$$k_1 = \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} \text{ N/mm} \quad \text{at } x_1 = 0$$

$$k_2 = \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} \text{ N/mm} \quad \text{at } x_2 = 1$$

$$k_3 = \begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} \text{ N/mm} \quad \text{at } x_3 = 2$$

Using the procedure above, write the global stiffness matrix

$$K = \begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 20 \end{pmatrix} \text{ N/mm}$$

Global force vector

Equation (1) represents the nodal forces

$$\begin{pmatrix} 10 & 0 & 0 & 0 \\ -20 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

$$F = 0$$

 In equation (1), $u_1 = u_2 = u_3 = u_4 = 0$ because of the fixed supports at nodes 1, 2 and 3.

$$\begin{pmatrix} 10 & 0 & 0 & 0 \\ -20 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Eq. (2) yields

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{at } x_3 = 2$$

(i) From the P and F matrices for k , we get the nodal values:

$$U_1 = 0, U_2 = 0.0017 \text{ m}$$

$$U_3 = 0.0016, U_4 = 0.0011 \text{ m}$$

(ii) The FE system for element (1) is

$$\begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \end{Bmatrix}$$

Let $U_1 = 0 = U_1^{\text{known}}$. The nodal values for the element

$$\begin{aligned} U_2 &= U_1 + \frac{100}{100} (U_2 - U_1) \\ &= 0 + \frac{100}{100} (0.0017) \\ &= 0.0017 \text{ m} \end{aligned}$$

The spring stiffness is 10^6 N/m. k is the stiffness of the spring. U_1 and U_2 are the nodal values. F_1 and F_2 are the nodal forces. $F_1 = 0$ and $F_2 = 100$ N. The nodal values are $U_1 = 0$ and $U_2 = 0.0017$ m. The nodal forces are $F_1 = 0$ and $F_2 = 100$ N. The nodal values are $U_1 = 0$ and $U_2 = 0.0017$ m. The nodal forces are $F_1 = 0$ and $F_2 = 100$ N.



Notes

Complete the table for each order n and k by suit appropriate basis functions

Using a spreadsheet software or otherwise, verify the functions listed in the table below

$$[P^1] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad \text{(corresponding degrees of freedom: } U_1 \text{ and } U_2 \text{)}$$

$$[P^2] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad \text{(corresponding degrees of freedom: } U_1 \text{ and } U_2 \text{)}$$

$$[P^3] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad \text{(corresponding degrees of freedom: } U_1 \text{ and } U_2 \text{)}$$

$$[P^4] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad \text{(corresponding degrees of freedom: } U_1 \text{ and } U_2 \text{)}$$

$$[P^5] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{matrix} U_1 \\ U_2 \end{matrix} \quad \text{(corresponding degrees of freedom: } U_1 \text{ and } U_2 \text{)}$$

The assembly stiffness matrix of the entire structure of the system of nodes can be obtained as

$$\mathbf{K} = \begin{bmatrix}
 K_1 & 0 & 0 & 0 \\
 0 & -(K_1 + K_2) & 0 & 0 \\
 0 & 0 & K_1 + K_2 + K_3 & 0 \\
 0 & 0 & 0 & -(K_2 + K_3)
 \end{bmatrix}
 \begin{bmatrix}
 U_1 \\
 U_2 \\
 U_3 \\
 U_4
 \end{bmatrix}$$

where the global stiffness matrix is diagonal and symmetric as required

•

Global system of the nodes is dependent degree of freedom of the entire structure of the system of nodes can be expressed as

$$\vec{F} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ F \end{Bmatrix}, \quad \vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

where F is the total force (external force) applied at node 4

•

Substituting the given geometry into the stiffness matrix yields:

$$[K]U = F$$

By applying the right boundary condition $u_4 = 0$ to the stiffness matrix, we obtain the following system of equations:

$$k^0 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 0 \\ 1000 \end{bmatrix}$$

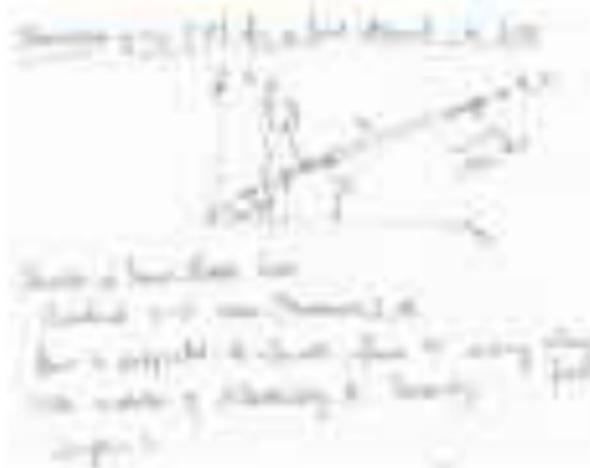
By applying the same boundary condition $u_4 = 0$, by setting the row and column corresponding to u_4 to 0, we obtain the following system of equations:

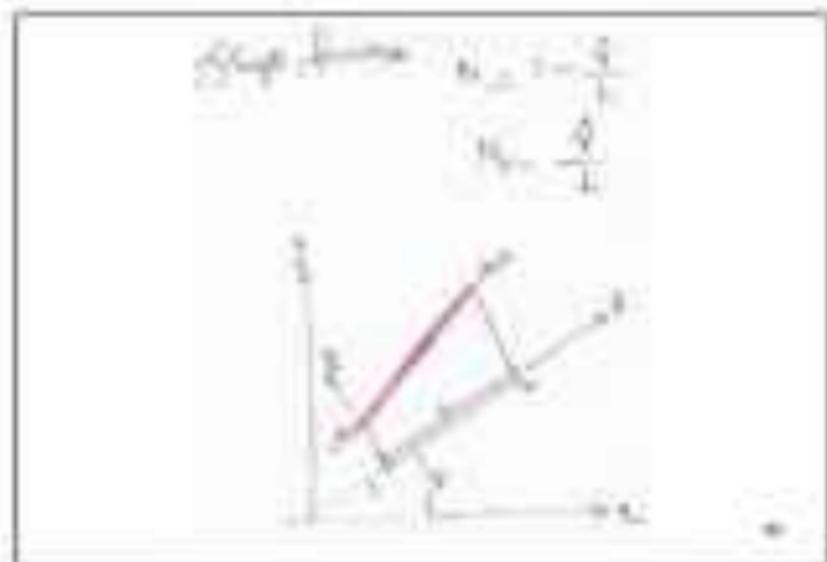
$$k^0 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1.11 \\ 0.47 \\ 1.79 \end{bmatrix} \times 10^{-3} \text{ m}$$

Axial Element

BAR ELEMENT (Linear Elastic Bar)





with a rotation

$$\begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix}$$

(1) (2) (3)

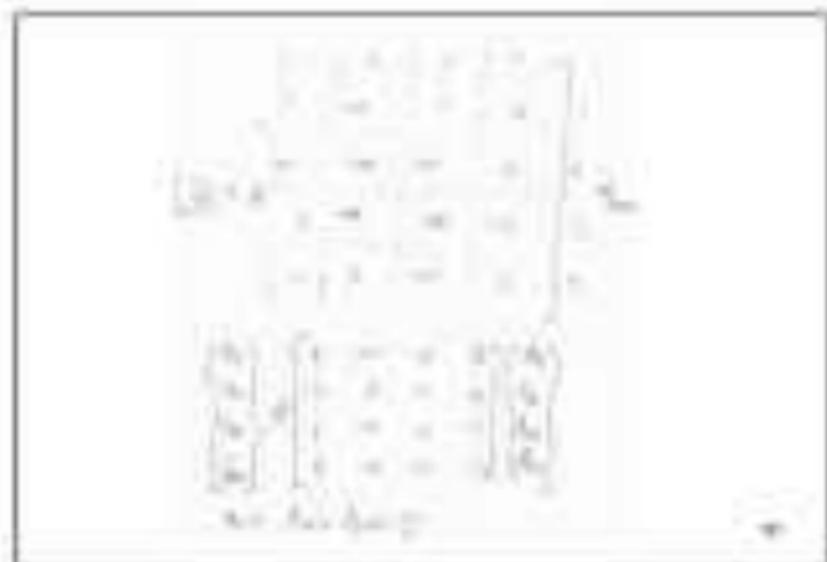
After this $q = \text{bar}$ $\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$



Load function $q(x)$

$$q(x) = \frac{2P}{L} \left(1 - \frac{4x}{L} \right)$$

$$u(x) = \frac{P}{6EI} \left(\frac{L^3}{8} - \frac{Lx^3}{2} + \frac{3L^2x^2}{4} - \frac{Lx^3}{2} \right)$$



$$C_{11} = \frac{EA}{L} \int_0^L \left(\frac{dN}{dx} \right)^2 dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{d}{dx} \left(\frac{1}{L} \left(\frac{x}{L} - 1 \right) \right) \right)^2 dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{1}{L} \right)^2 dx = \frac{EA}{L} \left(\frac{1}{L} \right) \int_0^L dx = \frac{EA}{L^2} \int_0^L dx$$

$$= \frac{EA}{L^2} \left[x \right]_0^L = \frac{EA}{L^2} \left[L - 0 \right] = \frac{EA}{L}$$

$$C_{22} = \frac{EA}{L} \int_0^L \left(\frac{dN}{dx} \right)^2 dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{d}{dx} \left(\frac{1}{L} \left(\frac{x}{L} \right) \right) \right)^2 dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{1}{L} \right)^2 dx = \frac{EA}{L} \left(\frac{1}{L} \right) \int_0^L dx = \frac{EA}{L^2} \int_0^L dx$$

$$= \frac{EA}{L^2} \left[x \right]_0^L = \frac{EA}{L^2} \left[L - 0 \right] = \frac{EA}{L}$$

$$C_{12} = \frac{EA}{L} \int_0^L \left(\frac{dN_1}{dx} \right) \left(\frac{dN_2}{dx} \right) dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{d}{dx} \left(\frac{1}{L} \left(\frac{x}{L} - 1 \right) \right) \right) \left(\frac{d}{dx} \left(\frac{1}{L} \left(\frac{x}{L} \right) \right) \right) dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{1}{L} \right) \left(\frac{1}{L} \right) dx = \frac{EA}{L} \left(\frac{1}{L} \right) \int_0^L dx = \frac{EA}{L^2} \int_0^L dx$$

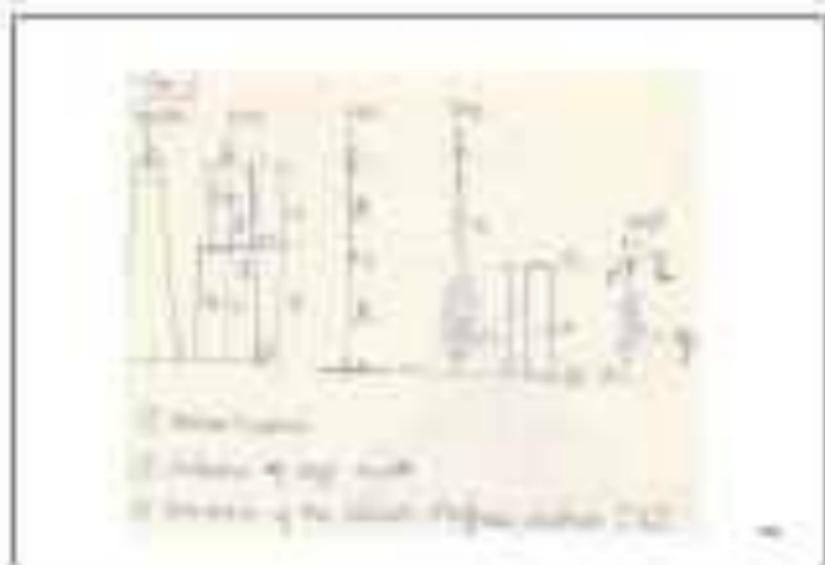
$$= \frac{EA}{L^2} \left[x \right]_0^L = \frac{EA}{L^2} \left[L - 0 \right] = \frac{EA}{L}$$

$$C_{21} = \frac{EA}{L} \int_0^L \left(\frac{dN_2}{dx} \right) \left(\frac{dN_1}{dx} \right) dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{d}{dx} \left(\frac{1}{L} \left(\frac{x}{L} \right) \right) \right) \left(\frac{d}{dx} \left(\frac{1}{L} \left(\frac{x}{L} - 1 \right) \right) \right) dx$$

$$= \frac{EA}{L} \int_0^L \left(\frac{1}{L} \right) \left(\frac{1}{L} \right) dx = \frac{EA}{L} \left(\frac{1}{L} \right) \int_0^L dx = \frac{EA}{L^2} \int_0^L dx$$

$$= \frac{EA}{L^2} \left[x \right]_0^L = \frac{EA}{L^2} \left[L - 0 \right] = \frac{EA}{L}$$



$$\begin{aligned}
 & \text{DAS} = \frac{Eh}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
 & \text{Q}_1 = \frac{Eh}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\
 & -\frac{Eh}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \frac{Eh}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

Stiffness of square plate DAS

$$\text{DAS} = \frac{Eh}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Q}_1 = \frac{Eh}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Apply BC: $u=0, v=0$ at $x=0, x=L$

Truss Analysis

Transformation of matrix is as



$$\begin{aligned}
 \text{global} \\
 k &= k_{xx}i + k_{yy}j \\
 \text{local} \\
 k &= k'_{xx}i' + k'_{yy}j'
 \end{aligned}$$

$$\begin{Bmatrix} k_x \\ k_y \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} k'_x \\ k'_y \end{Bmatrix} \quad \text{--- (1)}$$

Transformation Matrix

Ex: 1

Given: Find the area of the shaded region.

$$\begin{cases} y = 2x + 1 \\ y = -x + 4 \end{cases}$$

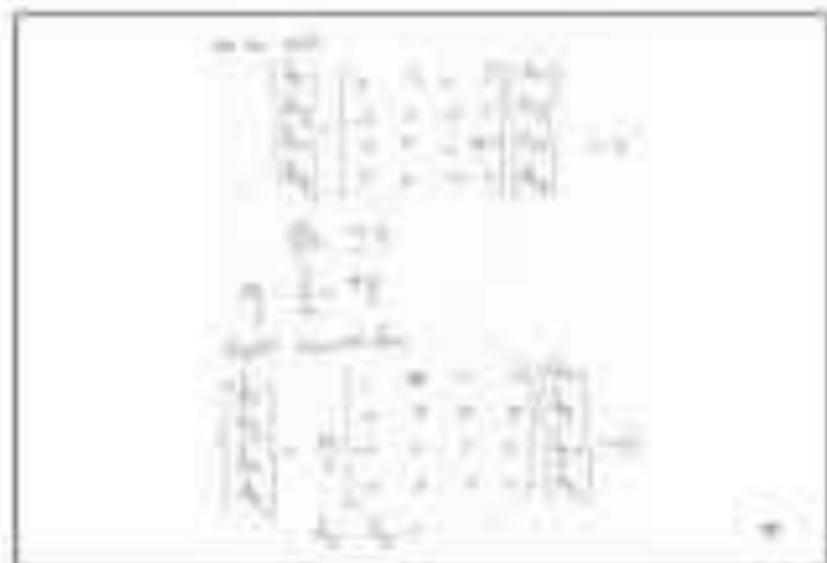
$$\begin{cases} 2x + 1 = -x + 4 \\ 3x = 3 \\ x = 1 \end{cases}$$

∴ Area of the shaded region =

$$\begin{aligned} & \int_0^1 (4 - x) dx - \int_0^1 (2x + 1) dx \\ & = \left[4x - \frac{x^2}{2} \right]_0^1 - \left[x^2 + x \right]_0^1 \\ & = \left(4(1) - \frac{1^2}{2} \right) - \left(1^2 + 1 \right) \\ & = \left(4 - \frac{1}{2} \right) - (2) \\ & = \frac{8}{2} - \frac{1}{2} - 2 \\ & = \frac{7}{2} - 2 \\ & = \frac{7}{2} - \frac{4}{2} \\ & = \frac{3}{2} \end{aligned}$$

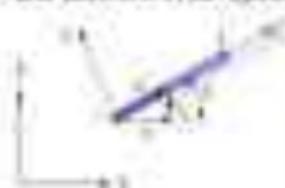
$\frac{1}{2} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \frac{1}{2} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx$
 $= \frac{1}{2} \left[\sin^{-1} x \right]_{-1}^1 = \frac{1}{2} (\sin^{-1} 1 - \sin^{-1} -1)$
 $= \frac{1}{2} (\frac{\pi}{2} - (-\frac{\pi}{2})) = \frac{1}{2} (\pi) = \frac{\pi}{2}$

$\frac{1}{2} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2}$



$$\begin{aligned}
 \text{Global } \mathbf{K} &= \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \\
 \text{Global } \mathbf{F} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

Bar Element in 2D Space



Node	Label
1, 2	2, 1
3, 4	4, 3
Global	Local

The local nodes of the bar element are:

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

The direction cosine of the bar element is:

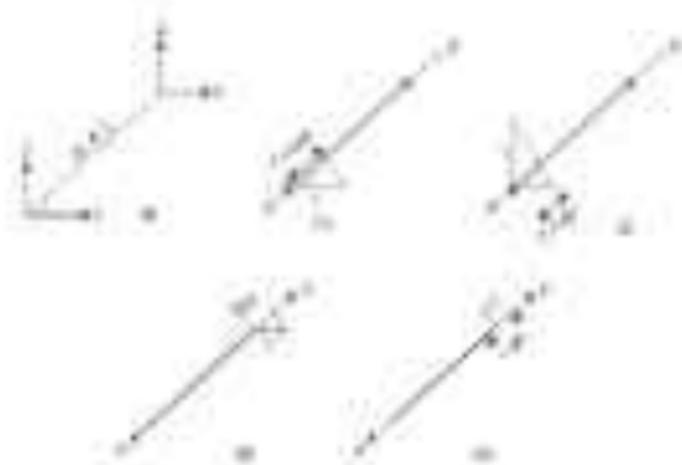
$$\mathbf{L} = \begin{bmatrix} L_x \\ L_y \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} L_x/L & L_y/L \\ -L_y/L & L_x/L \end{bmatrix}$$

Truss Element



$$d = \frac{P}{EA} \quad \rightarrow \quad f = \frac{EA}{L} d$$

$$\rightarrow \quad [K] = \frac{EA}{L} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



(i) The displacement of node 1 is a constant:

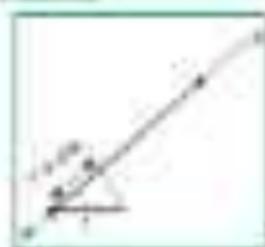
$$f = \frac{Eh}{L} \cos \theta$$

$$k_{11} = f \cos \theta = \frac{Eh}{L} \cos^2 \theta$$

$$k_{12} = f \sin \theta = \frac{Eh}{L} \cos \theta \sin \theta$$

$$k_{13} = f \cos \theta = \frac{Eh}{L} \cos^2 \theta$$

$$k_{14} = f \sin \theta = \frac{Eh}{L} \cos \theta \sin \theta$$



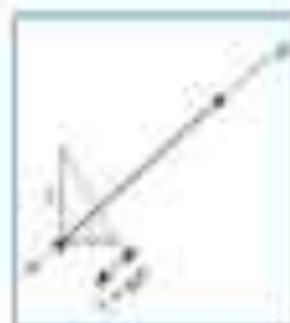
(ii) The displacement is constant in direction 2:

$$k_{21} = f \cos \theta = \frac{Eh}{L} \cos^2 \theta$$

$$k_{22} = f \sin \theta = \frac{Eh}{L} \cos \theta \sin \theta$$

$$k_{23} = -f \cos \theta = -\frac{Eh}{L} \cos^2 \theta$$

$$k_{24} = -f \sin \theta = -\frac{Eh}{L} \cos \theta \sin \theta$$



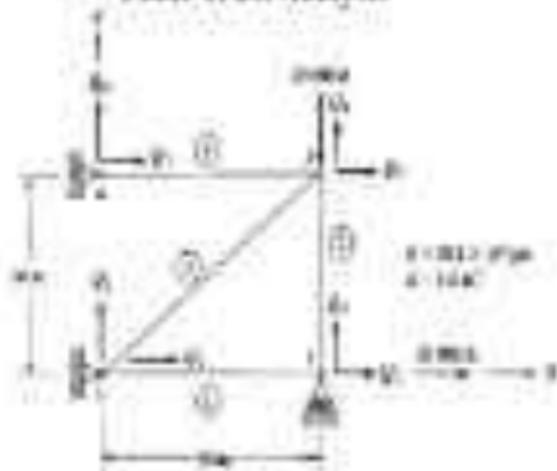
∴ The stiffness matrix is

$$[k] = \frac{EA}{L} \begin{bmatrix} -\cos^2\theta & \cos\theta\sin\theta & \cos\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & \sin\theta\cos\theta & -\sin^2\theta \\ \cos\theta & \sin\theta\cos\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

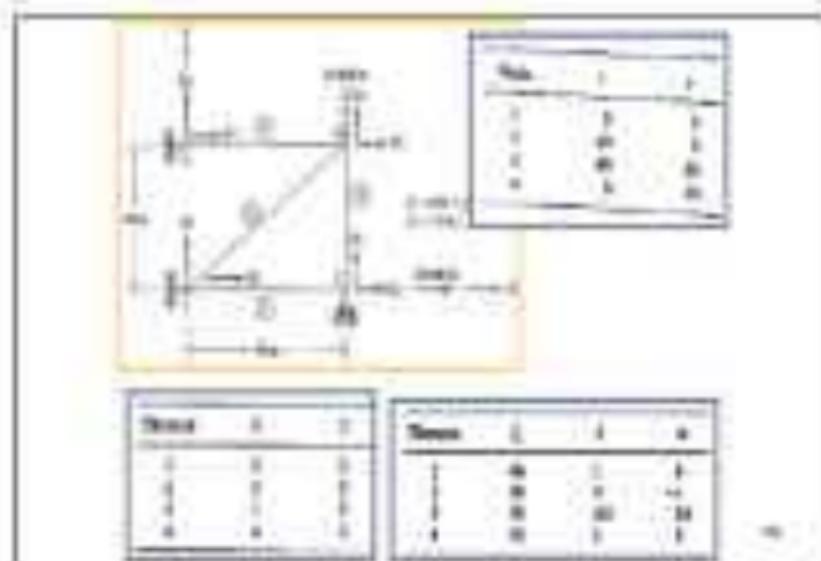
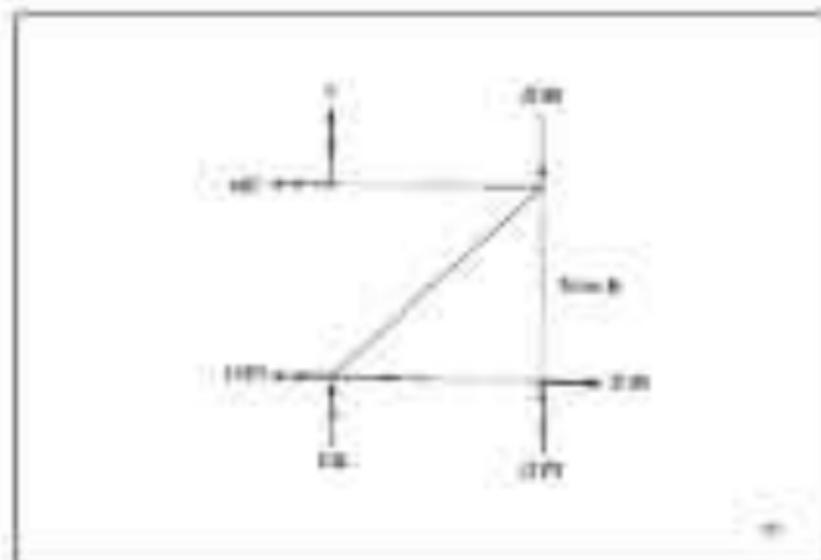
$$[k] = \frac{EA}{L} \begin{bmatrix} C^2 & CS & C & -CS \\ CS & S^2 & S & -S^2 \\ C & S & C^2 & -CS \\ -CS & -S^2 & -S & S^2 \end{bmatrix}$$

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Plane Truss Analysis



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Element Stiffness Matrix

$$k = \frac{E_s A_s}{L} \begin{bmatrix} \ell & m & -\ell & -m \\ m & n & -m & -n \\ -\ell & -m & \ell & m \\ -m & -n & m & n \end{bmatrix}$$

$$\ell = \cos \theta$$

$$m = \cos \phi$$

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$$k^e = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \text{Global dof} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$k^f = \frac{29.5 \times 10^6}{30} \begin{bmatrix} 5 & 6 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{array}{l} 5 \\ 6 \\ 3 \\ 4 \end{array}$$

11

$$\mathbf{K}^1 = \frac{29.5 \times 10^6}{30} \begin{bmatrix}
 1 & 2 & 3 & 4 \\
 64 & 48 & -64 & -48 \\
 48 & 36 & -48 & -36 \\
 -64 & -48 & 64 & 48 \\
 -48 & -36 & 48 & 36
 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\mathbf{K}^2 = \frac{29.5 \times 10^6}{40} \begin{bmatrix}
 7 & 8 & 9 & 10 \\
 1 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 \\
 -1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0
 \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 9 \\ 10 \end{matrix}$$

$$\mathbf{K} = \frac{29.5 \times 10^6}{300} \begin{bmatrix}
 1 & 2 & 3 & 4 & 7 & 8 & 9 & 10 \\
 214 & 57 & -21 & 0 & -148 & -57 & 0 & 0 \\
 176 & 42 & 0 & 0 & -176 & -42 & 0 & 0 \\
 -148 & 0 & 148 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 36 & 0 & -36 & 0 & 0 \\
 -748 & -57 & 0 & 0 & 2148 & 57 & -148 & 0 \\
 57 & -42 & 0 & -21 & 176 & 42 & 0 & 0 \\
 0 & 0 & 0 & 0 & -148 & 0 & 148 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

$$\frac{26.5 \times 10^6}{90} \begin{bmatrix} 17 & 0 & 0 \\ 0 & 11.60 & 1.75 \\ 0 & 1.75 & 14.32 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 20000 \\ 0 \\ -22000 \end{Bmatrix}$$

Solution of these equations yields the displacements

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 27.21 \times 10^{-4} \\ 3.05 \times 10^{-4} \\ -22.25 \times 10^{-4} \end{Bmatrix} \text{ m}$$

The nodal displacement values for the entire structure are illustrated in the figure below

$$Q = \{0.02721 \times 10^{-2}, 0.305 \times 10^{-2}, -22.25 \times 10^{-4} \times 0\}^T \text{ m}$$

$$e = \frac{\delta}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e_1 = \frac{27.21 \times 10^{-4}}{0.1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 2721 \mu\text{m}$$

The strain is constant throughout

$$e_2 = \frac{3.05 \times 10^{-4}}{0.1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 305 \mu\text{m}$$

$$e_3 = -1220 \mu\text{m}$$

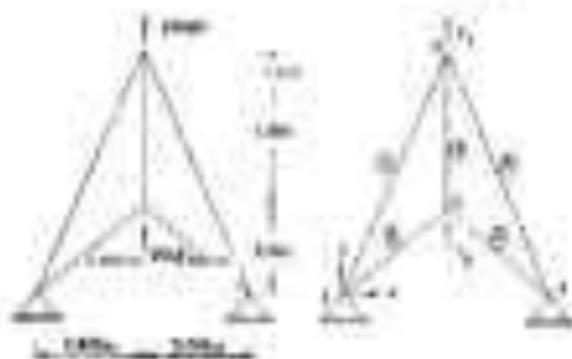
$$e_4 = 607 \mu\text{m}$$

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \frac{20 \times 10^3}{80} \begin{bmatrix} 110 & 70 & -10 & 0 & -100 & -100 & 0 & 0 \\ 70 & 110 & 0 & 0 & -100 & -100 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 0 & -100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 21.42 \times 10^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \begin{pmatrix} -15000.0 \\ 7100.0 \\ 21070.0 \\ -41070.0 \\ 0 \end{pmatrix} \text{ N}$$

Assume the area of cross section of all the members is the same. $E = 2 \times 10^8 \text{ N/m}^2$. Find the beam in the vertical direction.

$$\Delta = \frac{W}{E A}$$



Member 1 (0, 0, 0, 0, 0, 0, 1) = 100k

$$C_1 = 0.88, C_2 = 0.5 \quad \theta = 30^\circ$$

$$\begin{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix} \end{matrix}$$

$$[K] = \frac{EA}{100} \begin{bmatrix} 1.0 & 0.25 & -1.5 & -0.25 \\ 0.25 & 1.0 & -0.25 & -1.5 \\ -1.5 & -0.25 & 1.5 & 0.25 \\ -0.25 & -1.5 & 0.25 & 1.5 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

4

Member 2 (0, 0, 0, 0, 0, 0, 1) = 100k

$$C_1 = 0.88, C_2 = 0.5, \theta = 30^\circ$$

$$\begin{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix} \end{matrix}$$

$$[K] = \frac{EA}{100} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

4

Member 4 (200, 0, 0), (175, 0, 0)
 $L = 25\text{cm}$, $C_x = 0.4$, $C_y = -0.3$ $\theta = 30^\circ$

$$[k_4] = \frac{EA}{1000} \begin{matrix} & \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 7.5 & -4.5 & -7.5 & 4.5 \\ -4.5 & 3 & 4.5 & -3 \\ -7.5 & 4.5 & 7.5 & -4.5 \\ 4.5 & -3 & -4.5 & 3 \end{bmatrix} & \begin{matrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \end{matrix} \end{matrix}$$

Member 5 (0, 0, 0), (175, 0, 0)
 $L = 175\text{cm}$, $C_x = 0.5$, $C_y = -0.288$ $\theta = 30^\circ$

$$[k_5] = \frac{EA}{1000} \begin{matrix} & \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1.44 & -2.50 & -1.44 & 2.50 \\ -2.50 & 4.33 & 2.50 & -4.33 \\ -1.44 & 2.50 & 1.44 & -2.50 \\ 2.50 & -4.33 & -2.50 & 4.33 \end{bmatrix} & \begin{matrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \end{matrix} \end{matrix}$$

the stiffness matrix for the structure is given by,

$$[K] = \frac{EA}{1000} \begin{bmatrix} 18.0 & -11.0 \\ -11.0 & 15.0 \end{bmatrix}$$

The nodal load vector is

$$\{P\} = \begin{Bmatrix} 100.0 \\ 50.0 \end{Bmatrix}$$

Hence, the equilibrium equation is,

$$\frac{EA}{1000} \begin{bmatrix} 18.0 & -11.0 \\ -11.0 & 15.0 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} 100.0 \\ 50.0 \end{Bmatrix}$$

Solving the above equations we get,

$$r_2 = \frac{0.5872}{\Delta}$$

$$r_1 = \frac{0.5558}{\Delta}$$

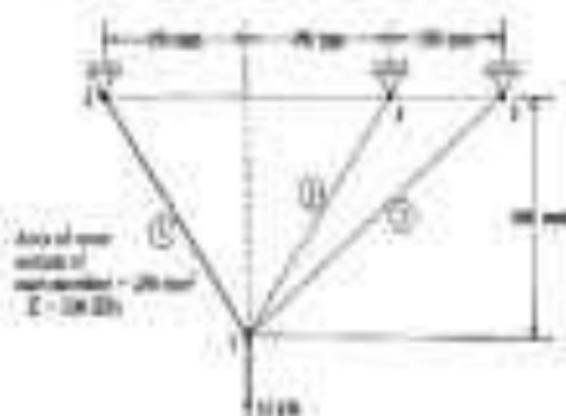
The nodal force in the member 2 along the global direction at its joints (Fig. 7.2) may be found out as

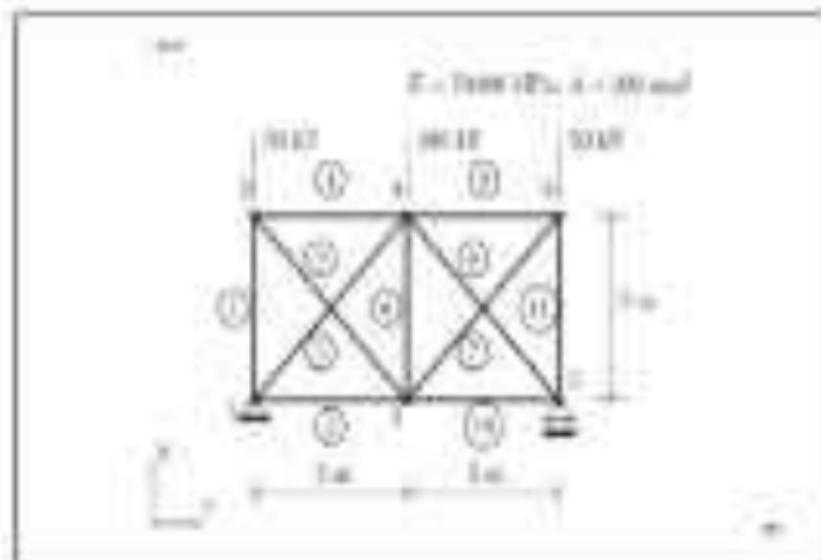
$$\{F\} = \{k\} \{d\}$$

$$= \frac{EA}{300} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001174 \\ 0 \\ 0.002347 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.174 \\ 0 \\ -1.174 \end{Bmatrix}$$

Thus, the axial force in the member 2 is 1.174 kN (tension).

Class Exercise





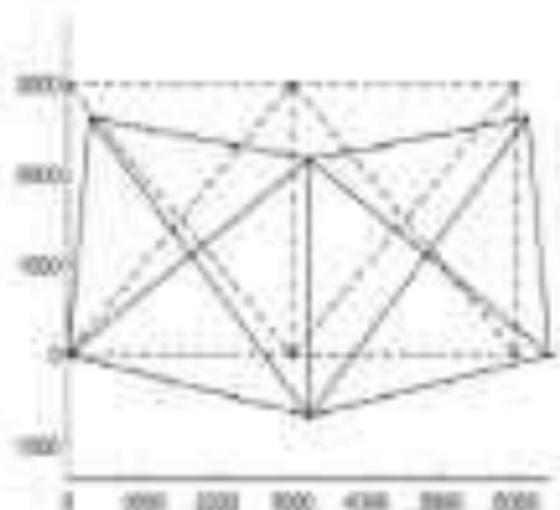
MEMBER NO.	DISPLACEMENTS
MEM NO.	MEM NO.
-118.8833	1.0000 4
103.4219	2.0000 4
-62.8875	3.0000 7.1429
+44.3380	4.0000 -0.3380
-173.5487	5.0000 5.2875
+66.8877	6.0000 -02.8860
-62.8875	7.0000 6.8875
-173.5487	8.0000 -01.8860
+44.3380	9.0000 4
103.4219	10.0000 5.2875
-118.8833	11.0000 -0.3380

Reactions

and =

$1.0e+05 *$

0.0000	0.0000
0.0000	1.0000
0.0001	1.0000



Displacements



$$Q_1 = 0.026517 \text{ cm}$$

$$Q_2 = 0.038811 \text{ cm}$$

$$Q_3 = 0.347903 \text{ cm}$$

$$Q_4 = -0.360013 \text{ cm}$$

THREE-DIMENSIONAL TRUSSES

$$k = \frac{E_c A_c}{L_c} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix}$$

Formulation of Shape/Interpolation Function

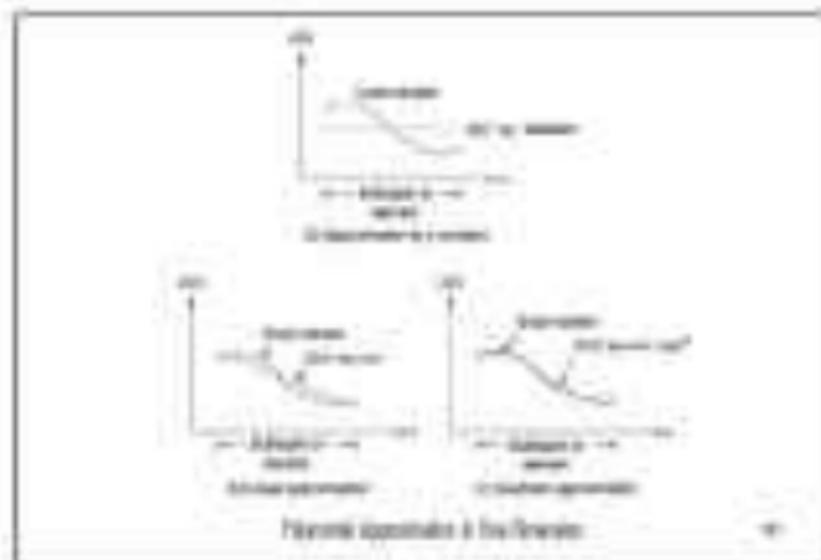


Interpolation Models

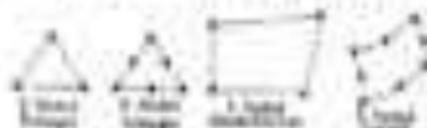
The function used to represent the behavior of the solution within an element are called interpolation functions or approximating functions or interpolation models. The following are standard functions that have been used widely used by the literature that are of the following classes:

1. It is easier to formulate and compute the finite element equations with polynomial type interpolation functions. Specifically, it is hard to perform differentiations or integrations with polynomials.
2. It is possible to improve the accuracy of the results by increasing the order of the polynomials, as shown in Figure 1.1. Theoretically, a polynomial of infinite order converges to the exact solution. But in practice, we use polynomials of finite order only as an approximation.

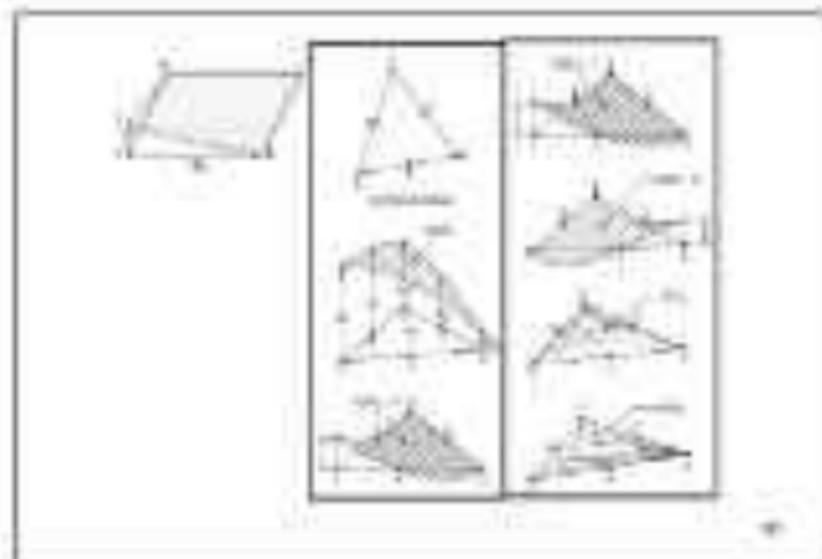




- Approximate polynomial $P(x)$ with n – **linear element**
- Approximate polynomial with n – **higher order element**



- **FE & Grid**... the discretization of the region can be improved by two methods:
 - **h method**: the location of the nodes can change without changing the total number of elements.
 - **p method**: number of nodes is increased.
- To improve the accuracy of problem, increase the order of polynomial called **p method**.



Guidelines to selecting approximate functions for displacements

1. Polynomials are commonly used approximate functions. Polynomials are chosen that give the most accurate results for a given order of displacement.

$$u = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 xy^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^3 + a_{10} y^3 + a_{11} x^2 y^2 + a_{12} xy^3 + a_{13} x^3 y + a_{14} x^2 y^2 + a_{15} xy^3 + a_{16} x^3 y^2 + a_{17} x^2 y^3 + a_{18} x^3 y^3$$

2. The approximate function should be continuous within the element. Therefore, the displacement must be continuous across adjacent elements. This is not the case for the polynomial functions and hence because of the continuity and smooth variation of u .

1. The approximate function should provide a convenient continuity for all DDF at each node for elements (by element let denote) at all group common boundaries (not just interfaces for 2D and 3D elements).

For this reason, defining the global continuity in two or more elements usually comes in three patterns (see below):



The linear function around the element e and f remains identical. The requirement of continuity is relaxed, but equal to the displacement at both nodes for continuity.

is:

$$d_{2e}^{(1)} = d_{2f}^{(1)}$$

(The linear function is used carrying or connecting to each function it creates. The relative line both of nodes only will be identical)

1. The approximation function is modified by (1) adding degrees of freedom and for a class of constant strain within the element.



When a function satisfies the above conditions, it is called a shape function.

The function is called a complete function if it satisfies the above conditions.

This completeness condition is necessary for convergence to the exact answer.



The idea that the temperature function must allow for 100% means that the function must capable of yielding constant value.

$$\begin{aligned}
 & \text{if } T(x) = T_0 \text{ then } \frac{dT}{dx} = 0 \\
 & \text{and } T(x) = T_0 \text{ is a constant function.} \\
 & \text{The function } T(x) = T_0 \text{ is a constant function.} \\
 & \text{where } T_0 = \frac{1}{2}(T_1 + T_2) \text{ is a constant function.} \\
 & \text{The function } T(x) = T_0 \text{ is a constant function.} \\
 & \text{The function } T(x) = T_0 \text{ is a constant function.}
 \end{aligned}$$

Example 1: Find $T(x)$

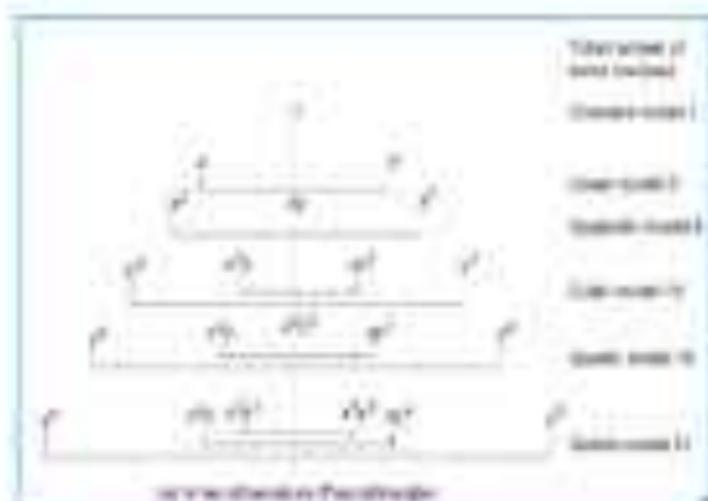
$$\begin{aligned}
 & \text{if } T(x) = T_0 + kx \\
 & \text{then } \frac{dT}{dx} = k
 \end{aligned}$$

It shows that the slope of the temperature function is constant and the only way to satisfy this condition is to have a constant value for $T(x)$ and not a function of x .

SELECTION OF THE ORDER OF THE INTERPOLATION POLYNOMIAL

While choosing the order of the polynomial, it is recommended to consider the following considerations before or to allow the system:

1. The interpolation polynomials should extend, as far as possible, the average requirements stated in Section 1.1.
2. The order of values of the load must be smaller than the polynomial order should be independent of the local mesh refinement.
3. The number of prescribed conditions (all) should be equal to the number of nodes (degree of freedom) of the element (n).



$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$u_1 = u_1$
 $u_2 = u_2$

$$\boxed{u_1 = [K] u_2}$$

$$u_2 = [K]^{-1} u_1 \rightarrow \text{inverse}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [K]^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\boxed{u_1 = [K]^{-1} u_2}$$

$$\text{with } [K]^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore [K] = [K]^{-1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_0^1 \left[\begin{matrix} 1 \\ -\frac{x}{l} \\ \frac{x}{l} \end{matrix} \right] \rightarrow \text{shape function}$$

$$N_1 = 1 - \frac{x}{l} \quad \text{and} \quad N_2 = \frac{x}{l}$$



1D stress line element $\xrightarrow{u_0}$

$$u = N_1 u_1 + N_2 u_2 \quad \text{--- (1)}$$

Displacement = u

$$\text{Node } 1: \quad \begin{matrix} u_1 \\ \vdots \\ u_2 \end{matrix}$$

$$\{u\} = [N] \{U\} \quad \text{--- (2)}$$

$N_1 = 1 - \frac{x}{L} + \frac{x^2}{L^2}$	
$N_2 = -\frac{x}{L} + \frac{x^2}{L^2}$	
$N_3 = \frac{x}{L} - \frac{x^2}{L^2}$	

Apply with $q = \text{const.}$
Verify

Development of stiffness matrix

(1D) $N_1 u_1 + N_2 u_2 + N_3 u_3$

$$\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$\Rightarrow \mathbf{K} \mathbf{U} = \mathbf{F}$

$$c = \frac{\partial u}{\partial x} \quad \text{PK-1014}$$

$$\left[-\frac{1}{2} + \frac{2x}{l} \quad \frac{2x}{l} - 1 \quad \frac{1}{2} - \frac{2x}{l} \right] \text{PK}$$

$$E = [E] [B] [u]$$

$$E_{el} = \int_V [B]^T [E] [B] dV$$

$$\text{PK-1014} \quad \int_{-l/2}^{l/2} \begin{bmatrix} 1 & 2x/l \\ 2x/l & 1 \end{bmatrix} [E] \begin{bmatrix} -1/2 + 2x/l & 2x/l - 1 & 1/2 - 2x/l \end{bmatrix} dx$$

PK-1014

PK-1014

$$E_{el} = \int_{-l/2}^{l/2} \begin{bmatrix} 1 & 2x/l \\ 2x/l & 1 \end{bmatrix} [E] \begin{bmatrix} -1/2 + 2x/l & 2x/l - 1 & 1/2 - 2x/l \end{bmatrix} dx$$

PK-1014

$$\begin{aligned}
 k_{11} &= \left(-\frac{1}{2} + \frac{2x}{l}\right) \left(-\frac{1}{2} + \frac{2x}{l}\right) \\
 &= \left[\frac{-x^2}{l^2} - \frac{-4x}{l} + \frac{1}{4}\right]
 \end{aligned}$$

$$\begin{aligned}
 k_{12} &= \left(-\frac{1}{2} + \frac{2x}{l}\right) \left(\frac{1}{2} - \frac{2x}{l}\right) \\
 &= \left[\frac{2x^2}{l^2} - \frac{2x}{l} + \frac{1}{4}\right]
 \end{aligned}$$

$$\begin{aligned}
 k_{22} &= \left(\frac{1}{2} - \frac{2x}{l}\right) \left(\frac{1}{2} - \frac{2x}{l}\right) \\
 &= \left[-\frac{2x^2}{l^2} + \frac{4x}{l} - \frac{1}{4}\right]
 \end{aligned}$$

$$\begin{aligned}
 k_{21} &= k_{12} \\
 &= \left(\frac{2x^2}{l^2} - \frac{2x}{l} + \frac{1}{4}\right) \\
 &= \left[\frac{2x^2}{l^2} - \frac{2x}{l} + \frac{1}{4}\right]
 \end{aligned}$$

$$\begin{aligned}
 k_{22} &= \left(\frac{1}{2} - \frac{2x}{l}\right) \left(\frac{1}{2} - \frac{2x}{l}\right) \\
 &= \left[-\frac{2x^2}{l^2} + \frac{4x}{l} - \frac{1}{4}\right]
 \end{aligned}$$

$$\begin{aligned}
 k_{33} &= \left(\frac{1}{2} - \frac{2x}{l}\right) \left(\frac{1}{2} - \frac{2x}{l}\right) \\
 &= \left[\frac{2x^2}{l^2} - \frac{4x}{l} + \frac{1}{4}\right]
 \end{aligned}$$

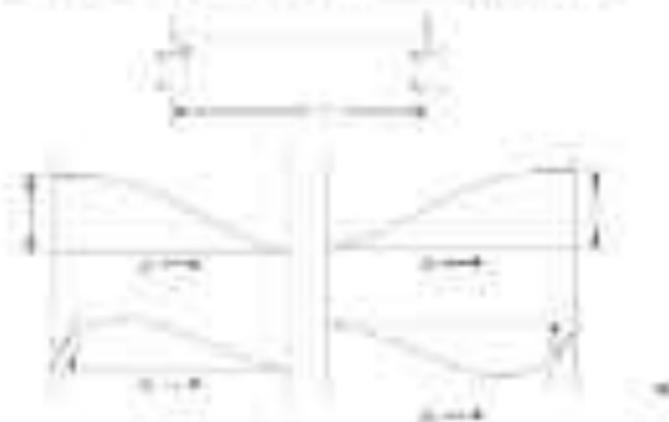
$$\vec{c} = \vec{F} = \begin{bmatrix} \frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} & \frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} & -\frac{3x^2}{2l} + \frac{3x}{l} - \frac{1}{2} \\ \frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} & \frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} & -\frac{3x^2}{2l} + \frac{3x}{l} - \frac{1}{2} \\ -\frac{3x^2}{2l} + \frac{3x}{l} - \frac{1}{2} & -\frac{3x^2}{2l} + \frac{3x}{l} - \frac{1}{2} & \frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned}
 D(x) &= \frac{3}{2l} \left(\frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} \right)^2 - \\
 &\quad \left[\frac{3x^2}{2l} - \frac{3x}{l} + \frac{1}{2} \right] \left[-\frac{3x^2}{2l} + \frac{3x}{l} - \frac{1}{2} \right] \\
 &= \frac{9x^4}{4l^2} - \frac{9x^3}{2l^2} + \frac{3x^2}{4l} - \frac{3x^2}{2l} + \frac{3x}{l} - \frac{1}{2} + \frac{3x^3}{2l} - \frac{3x^2}{l} + \frac{3x}{2} - \frac{1}{4} \\
 &= \frac{9x^4}{4l^2} - \frac{3x^2}{4l} - \frac{1}{4} \\
 D(x) &= \frac{9x^4}{4l^2}
 \end{aligned}$$

Final stiffness matrix

$$\begin{matrix}
 \text{Node} \\
 \text{[K]} = \frac{EA}{L}
 \end{matrix}
 \begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
 \end{bmatrix}$$

Using polynomial functions (generalized coefficients)
 Hermite shape functions for a 1D rod with fixed ends



To fit our C^1 -continuity we have to write:

$$\begin{aligned}
 \text{where} \\
 \left\{ \begin{array}{l} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \right\} & \quad w_1 = \frac{dy_1}{dx} \quad w_2 = \frac{dy_2}{dx}
 \end{aligned}$$

Now use the shape functions in order to find out the following:

$$v = N_1 + N_2 x + N_3 x^2 + N_4 x^3$$

$$w = \frac{dv}{dx} = w_1 + 2N_3 x + 3N_4 x^2$$

□

For translation we also need another matrix:

$v_1 = v$	$ \begin{aligned} v_3 &= N_1 + N_2 x + N_3 x^2 + N_4 x^3 \\ w_2 &= w_1 + 2N_3 x + 3N_4 x^2 \end{aligned} $
$v_2 = v$	
$w_1 = w$	
$w_2 = w$	

$$\left\{ \begin{array}{l} v_1 \\ v_2 \\ w_1 \\ w_2 \end{array} \right\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix} \left\{ \begin{array}{l} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \right\}$$

□

$$\begin{aligned}
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 & u = u_1 + u_2 + u_3 + u_4 \\
 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
 & = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
 & = [N_1 \ N_2 \ N_3 \ N_4] \cdot \{U\} = [N] \cdot \{U\}
 \end{aligned}$$

where:

$$[N] = [N_1 \ N_2 \ N_3 \ N_4]$$

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \quad N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

The shape functions are used to determine the displacement field $u(x)$ within the element. The displacement is given by:

$$u(x) = N_1 u_1 + N_2 \theta_1 + N_3 u_2 + N_4 \theta_2$$

where $u_1, \theta_1, u_2, \theta_2$ are the nodal displacements and rotations.

The shape functions are also used to determine the strain field $\epsilon(x)$ within the element. The strain is given by:

$$\epsilon(x) = \frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} \theta_1 + \frac{dN_3}{dx} u_2 + \frac{dN_4}{dx} \theta_2$$

The shape functions are also used to determine the stress field $\sigma(x)$ within the element. The stress is given by:

$$\sigma(x) = E \epsilon(x) = E \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} \theta_1 + \frac{dN_3}{dx} u_2 + \frac{dN_4}{dx} \theta_2 \right)$$

The shape functions are also used to determine the internal force field $f(x)$ within the element. The internal force is given by:

$$f(x) = \int_V \sigma(x) dx = E \int_V \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} \theta_1 + \frac{dN_3}{dx} u_2 + \frac{dN_4}{dx} \theta_2 \right) dx$$

The shape functions are also used to determine the external force field $F(x)$ within the element. The external force is given by:

$$F(x) = \int_V f(x) dx = E \int_V \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} \theta_1 + \frac{dN_3}{dx} u_2 + \frac{dN_4}{dx} \theta_2 \right) dx$$



Natural Coordinates in Two Dimensions



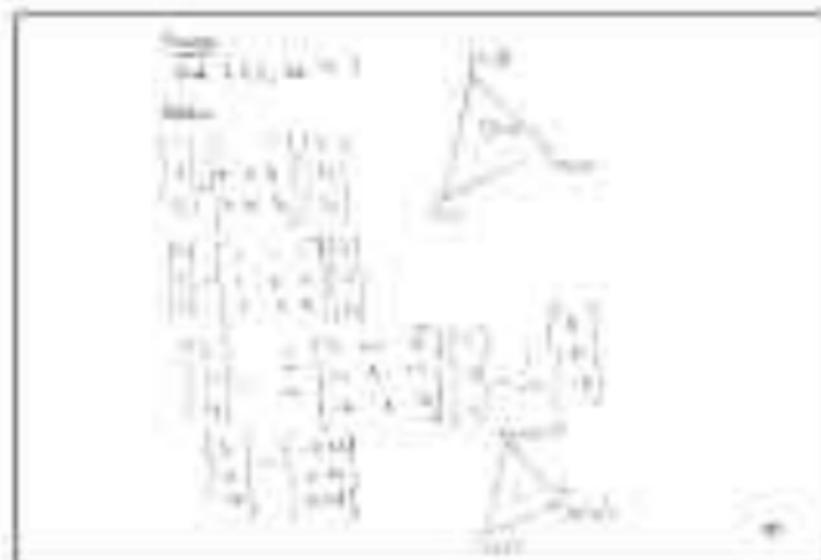
Types of natural coordinates

$$\begin{aligned} \xi_1 + \xi_2 + \xi_3 &= 1 \\ \eta_1 + \eta_2 + \eta_3 &= 0 \\ \zeta_1 + \zeta_2 + \zeta_3 &= 1 \end{aligned}$$

$$\begin{array}{ccc|ccc} \xi & \eta & \zeta & \xi_1 & \xi_2 & \xi_3 \\ \eta & \eta & \eta & \eta_1 & \eta_2 & \eta_3 \\ \zeta & \zeta & \zeta & \zeta_1 & \zeta_2 & \zeta_3 \end{array}$$



$$\begin{array}{ccc|ccc} \xi & \eta & \zeta & \xi_1 & \xi_2 & \xi_3 \\ \eta & \eta & \eta & \eta_1 & \eta_2 & \eta_3 \\ \zeta & \zeta & \zeta & \zeta_1 & \zeta_2 & \zeta_3 \end{array}$$



The closed form integration for the element:

$$\int_{\Omega} N_i^T N_j d\Omega = \frac{A}{3} \delta_{ij}$$

matrix $\mathbf{M} = \frac{A}{3} \mathbf{I}$

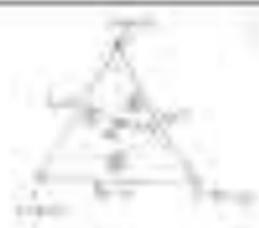
$$\mathbf{M}^{-1} = \frac{3}{A} \mathbf{I}$$

$$\int_{\Omega} N_i^T N_j d\Omega = \frac{A}{3} \delta_{ij} = \frac{1}{3} \int_{\Omega} N_i^T N_j d\Omega$$

matrix

$$\mathbf{M}^{-1} = \frac{3}{A} \mathbf{I}$$

$$\int_{\Omega} N_i^T N_j d\Omega = \frac{A}{3} \delta_{ij} = \frac{1}{3} \int_{\Omega} N_i^T N_j d\Omega$$



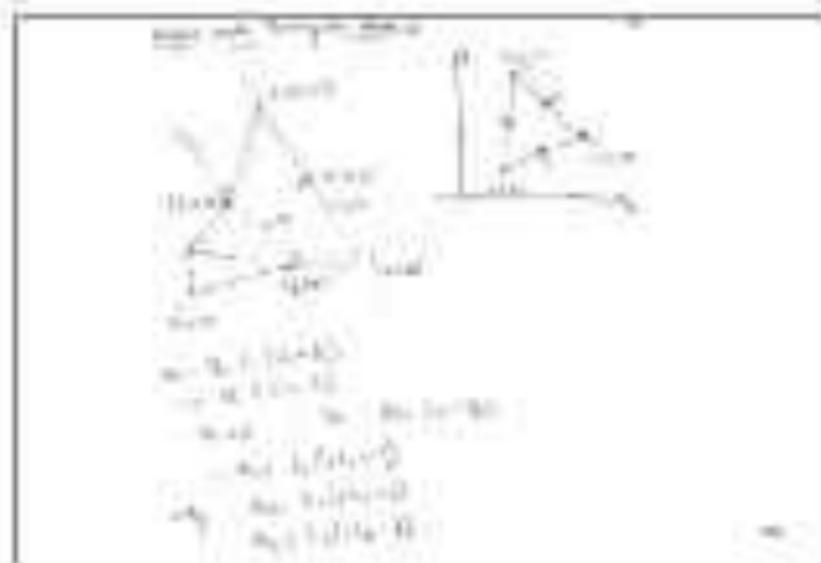
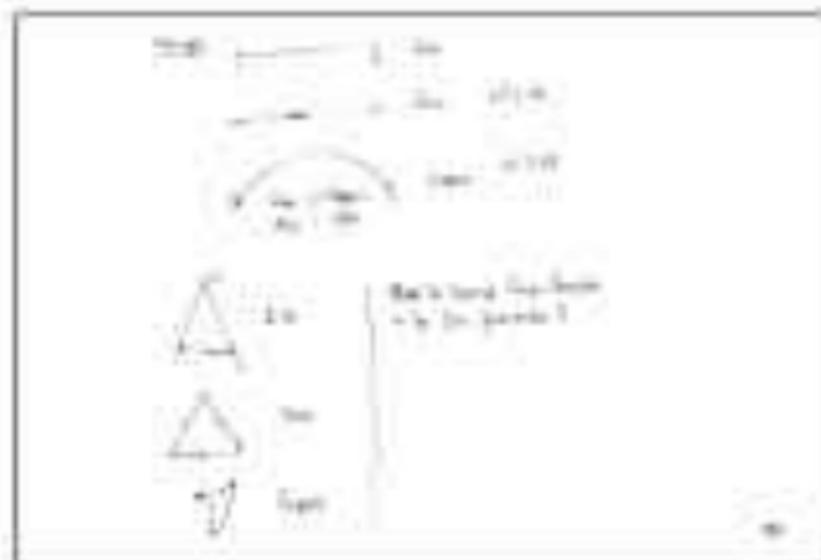
1. Node 1 - 1/3
 2. Node 2 - 1/3
 3. Node 3 - 1/3
 4. Midpoint of side 1-2 - 2
 5. Midpoint of side 1-3 - 3
 6. Midpoint of side 2-3 - 1

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

1. Node 1 - 1/3
 2. Node 2 - 1/3
 3. Node 3 - 1/3
 4. Midpoint of side 1-2 - 2
 5. Midpoint of side 1-3 - 3
 6. Midpoint of side 2-3 - 1

1. Node 1 - 1/3
 2. Node 2 - 1/3
 3. Node 3 - 1/3
 4. Midpoint of side 1-2 - 2
 5. Midpoint of side 1-3 - 3
 6. Midpoint of side 2-3 - 1

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$



$N_1 = \frac{1}{3} (2x + y)$
 $N_2 = \frac{1}{3} (2y + x)$
 $N_3 = \frac{1}{3} (2x + y)$
 $N_4 = \frac{1}{3} (2y + x)$

The element is divided into three sub-triangles by lines from each node to the midpoint of the opposite side.

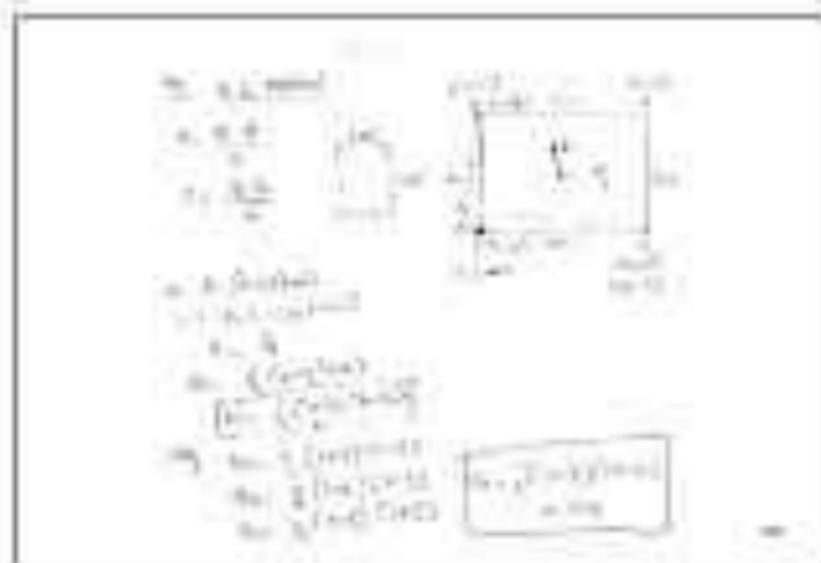
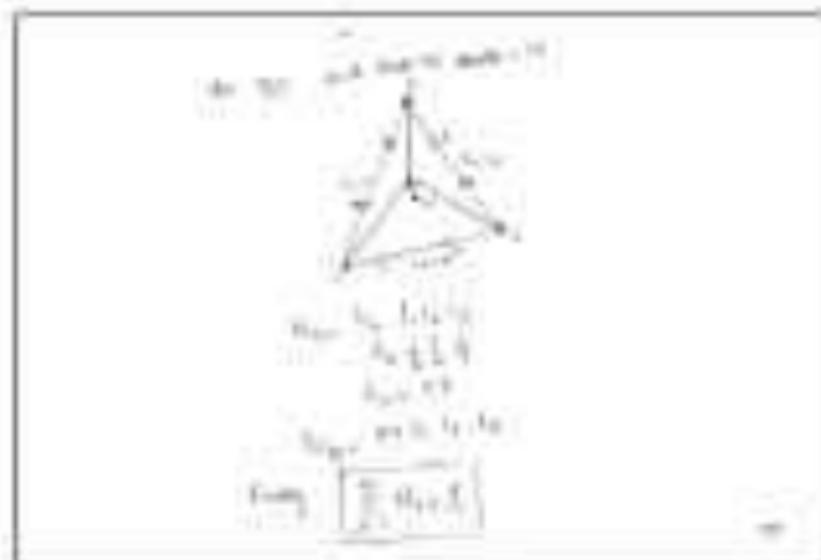
$N_1 = \frac{1}{3} (2x + y)$
 $N_2 = \frac{1}{3} (2y + x)$
 $N_3 = \frac{1}{3} (2x + y)$
 $N_4 = \frac{1}{3} (2y + x)$

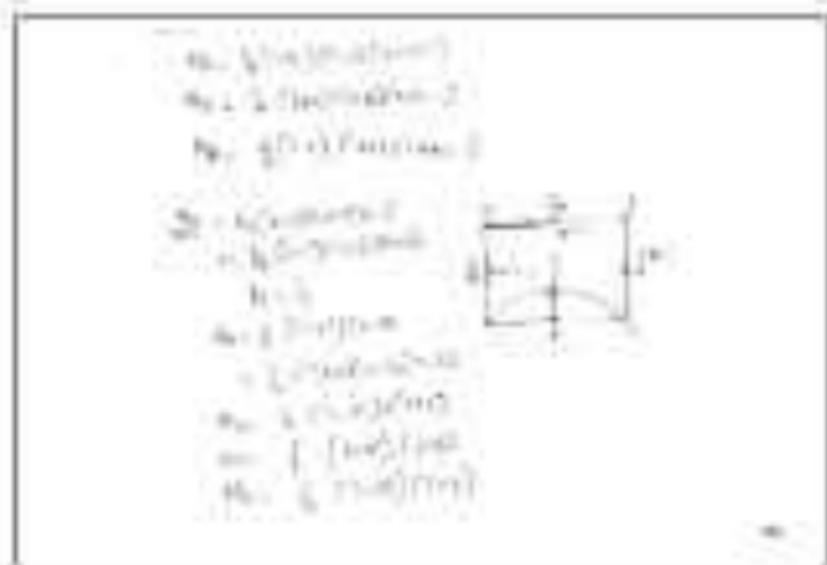
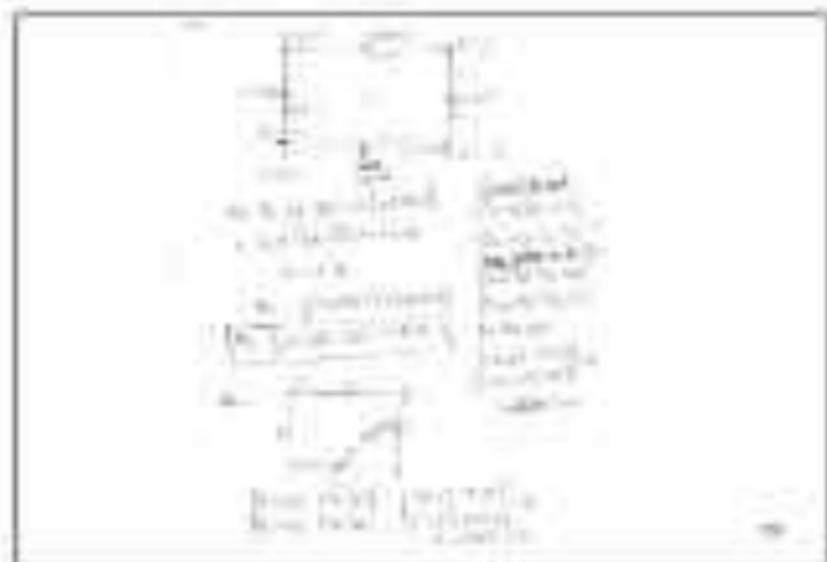
$\sigma = \frac{P}{A}$
 $\epsilon = \frac{\Delta L}{L}$
 $E = \frac{\sigma}{\epsilon}$
 $\Delta L = \frac{PL}{EA}$
 $\Delta L = \frac{P \cdot L}{E \cdot A}$
 $\Delta L = \frac{P \cdot L}{E \cdot \pi r^2}$
 $\Delta L = \frac{P \cdot L}{E \cdot \pi \left(\frac{D}{2}\right)^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$

$\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$

$\sigma = \frac{P}{A}$
 $\epsilon = \frac{\Delta L}{L}$
 $E = \frac{\sigma}{\epsilon}$
 $\Delta L = \frac{PL}{EA}$
 $\Delta L = \frac{P \cdot L}{E \cdot A}$
 $\Delta L = \frac{P \cdot L}{E \cdot \pi r^2}$
 $\Delta L = \frac{P \cdot L}{E \cdot \pi \left(\frac{D}{2}\right)^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$

$\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$
 $\Delta L = \frac{4PL}{E\pi D^2}$





1. $\frac{1}{2} \times \text{base} \times \text{height}$
 2. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 3. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$
 4. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 5. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

6. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 7. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

8. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 9. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$



10. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 11. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

12. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 13. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

1. $\frac{1}{2} \times \text{base} \times \text{height}$
 2. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 3. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$
 4. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 5. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$



6. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 7. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

8. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 9. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

10. $\frac{1}{2} \times \text{side} \times \text{side} \times \sin(\theta)$
 11. $\frac{1}{2} \times \text{side} \times \text{side} \times \cos(\theta)$

1. $u = 2x + y$
 2. $u = 3x + 2y$
 3. $u = 4x + 3y$
 4. $u = 5x + 4y$
 5. $u = 6x + 5y$
 6. $u = 7x + 6y$
 7. $u = 8x + 7y$
 8. $u = 9x + 8y$
 9. $u = 10x + 9y$
 10. $u = 11x + 10y$
 11. $u = 12x + 11y$
 12. $u = 13x + 12y$
 13. $u = 14x + 13y$
 14. $u = 15x + 14y$
 15. $u = 16x + 15y$
 16. $u = 17x + 16y$
 17. $u = 18x + 17y$
 18. $u = 19x + 18y$
 19. $u = 20x + 19y$
 20. $u = 21x + 20y$
 21. $u = 22x + 21y$
 22. $u = 23x + 22y$
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$N_1 = \frac{x_2 - x}{x_2 - x_1} = 1 - \frac{x - x_1}{x_2 - x_1}$
 $= 1 - \xi \frac{x_2 - x_1}{x_2 - x_1}$
 $= 1 - \xi$
 $N_2 = \frac{x - x_1}{x_2 - x_1}$

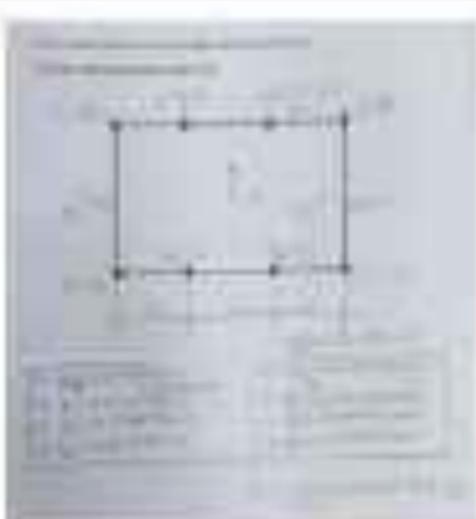


Type of element: 

Lagrange: 

Isoparametric: 

Note: $\xi = \frac{x - x_1}{x_2 - x_1}$



The slide shows a 2D element (a square) with nodes labeled 1, 2, 3, and 4. A coordinate system is shown with x and y axes. The element is divided into four quadrants by the x and y axes. The nodes are located at the corners of the square.

Convergence Requirements

1. The displacement function must be continuous within the element. This condition can easily be satisfied by choosing polynomials for the displacement field.

(This is why a meshing step is not a step change)

2. The displacement function must be capable of representing rigid body displacements of the element. That is when the nodes are given such displacements corresponding to a rigid body motion, the element should not experience any strain and hence leads to zero nodal loads. The constant terms in the polynomial used for the displacement models would usually meet the condition.

17

1. The displacement function must be capable of representing constant strain states within the element. The reason for this requirement can be understood if we imagine the condition when the body or structure is divided into smaller and smaller elements. As these elements approach infinitesimal size, the strains in each element also approach constant values. Hence the assumed displacement function should include terms for representing constant strain states. For one, two and three-dimensional elasticity problems, the linear terms present in the polynomial satisfy the requirement. However, in the case of beam, plate and shell elements, this condition will be referred to as 'constant curvature' instead of 'constant strain'. The 2D study of satisfying this condition in these situations would be discussed later.

18

Let $u(x)$ vary linearly with x as shown

$u(0) = u_1 = 0$

$u(L) = u_2 = 1$

$u(0) = u_1 = 0$

$u(L) = u_2 = 1$ or $u_1 = 0$ or $u_2 = 1$

$$u(x) = \frac{u_2 - u_1}{L} x + u_1$$

$$u(x) = \frac{u_2 - u_1}{L} x + u_1 = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2$$



$$u(x) = N_1(x) u_1 + N_2(x) u_2$$

$$u(x) = \sum_{i=1}^2 N_i(x) u_i$$

where N_i = Initial Shape Functions

$$u(x) = [N(x)] N_i(x) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$N = [N_1(x) \quad N_2(x)]$$

$$u(x) = N d$$

Shape Function Matrix
= Discretization

$\hat{u}(x)$ approximates $u(x)$.

$$\text{Error} = \hat{u}(x) - u(x)$$

Strains

$$\begin{aligned}
 \epsilon_x &= \frac{du(x)}{dx} \\
 &= \frac{d}{dx} u(x) \\
 &= \frac{d}{dx} [N_i(x) \cdot u_i] \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \\
 &= \mathbf{LN}_i^T \\
 &= \mathbf{B}_i \cdot \mathbf{d} \quad [\mathbf{B}(x) = \text{Strain Matrix}] \\
 &= \left[\frac{dN_i(x)}{dx} \quad \frac{dN_j(x)}{dx} \right] \begin{pmatrix} u_i \\ u_j \end{pmatrix} \\
 \boldsymbol{\epsilon} &= \mathbf{Bd}
 \end{aligned}$$

Constitutive Relation

$$\begin{aligned}
 \sigma_x &= E \epsilon_x \\
 \boldsymbol{\sigma} &= \mathbf{E} \boldsymbol{\epsilon}
 \end{aligned}$$

Minimum Potential Energy

The total potential energy is given by

$$\begin{aligned}
 \Pi &= U - W \\
 U &= U(x) - W(x) = U(x) - W(x)
 \end{aligned}$$

where U is strain energy stored in the body during deformation and

W is the work done by the external loads.

We have additional

$$\mathbf{N} = \mathbf{N}_i^T \quad \boldsymbol{\epsilon} = \mathbf{Bd} \quad \boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\epsilon}$$

with which, we have formulated the unknown nodal displacement

Variational Statement

$$\begin{aligned}
 \pi &= \int_{\Omega} (\sigma^T \cdot \epsilon) - \int_{\Omega} (\sigma^T \cdot \epsilon) - \int_{\Omega} \underbrace{(\sigma^T \cdot \epsilon)}_{\text{Lagrange multiplier}} \\
 &= \int_{\Omega} ((\sigma^T \cdot \epsilon) - \sigma^T \cdot \epsilon) - \int_{\Omega} (\sigma^T \cdot \epsilon) \\
 &= (A_1)^T \left(\int_{\Omega} (\sigma^T \cdot \epsilon) \right) - (A_1)^T \left(\int_{\Omega} (\sigma^T \cdot \epsilon) \right) \\
 &= (A_1)^T \left(\int_{\Omega} (\sigma^T \cdot \epsilon) \right) - \left(\int_{\Omega} (\sigma^T \cdot \epsilon) \right) \\
 &= \delta \pi = 2 \left(\int_{\Omega} (\sigma^T \cdot \epsilon) \right) - \left(\int_{\Omega} (\sigma^T \cdot \epsilon) \right) = 0
 \end{aligned}$$

$$= \delta \pi = 0$$

$$K = \int_{\Omega} \sigma^T \cdot \epsilon; P = \int_{\Omega} N^T \cdot p$$

$$K_1 = \left(1 - \frac{\nu}{2}\right); K_2 = \frac{\nu}{2}$$

$$\frac{dK_1}{dx} = -\frac{\nu}{2}; \frac{dK_2}{dx} = \frac{\nu}{2}; dx = 2 \cdot dx$$

$$A = \int_{\Omega} \left(\begin{bmatrix} 1 \\ \frac{\nu}{2} \\ \frac{\nu}{2} \\ 1 \end{bmatrix} dx \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) dx = \begin{bmatrix} \frac{dK}{dx} & -\frac{dK}{dx} \\ \frac{dK}{dx} & \frac{dK}{dx} \end{bmatrix}$$

Uniform Load

$$p = \int_0^L \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} q dx$$

Node values
 Element values

$$= \int_0^L \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} q dx$$

$$= \begin{pmatrix} qL \\ qL^2/2 \\ qL^3/3 \end{pmatrix}$$



Bar Element

1D bar element in 1D



$$\begin{aligned}
 & \text{Node values} \\
 & \text{Element values} \\
 & \text{Node values} \\
 & \text{Element values}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Node values} \\
 & \text{Element values} \\
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 & \text{Element values}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Node values} \\
 & \text{Element values} \\
 & \text{Node values} \\
 & \text{Element values}
 \end{aligned}$$

Write the local constitutive law

$$\sigma = E \varepsilon = E \frac{1}{L} (u_2 - u_1) \quad (10)$$

then

$$F = \sigma A = EA \frac{1}{L} (u_2 - u_1) \quad (11)$$

Insert this expression into the equilibrium

$$-F + F_0 = 0 \Rightarrow F = F_0$$

so

$$EA \frac{1}{L} (u_2 - u_1) = F_0 \quad (12)$$

Write a general expression

$$u_2 - u_1 = \frac{F_0 L}{EA} = \frac{F_0}{EA} L \quad (13)$$

Write the displacement of the element

$$u(x) = \frac{F_0}{EA} \left(\frac{x}{L} \right) \quad (14)$$

 at $x = 0$

Displacement

$$u(0) = 0 \quad (15)$$

Write the corresponding stress

$$\sigma(x) = E \varepsilon(x) = E \frac{1}{L} \frac{d}{dx} \left(\frac{F_0}{EA} x \right) \quad (16)$$

$$= \frac{F_0}{EA} \frac{E}{L} = \frac{F_0}{A} \quad (17)$$

Identification of stress and

the corresponding force

$$\sigma = \frac{F_0}{A} = \frac{F_0}{A} \frac{A}{A} = \frac{F_0}{A} \quad (18)$$

The corresponding force

$$F = \sigma A = F_0 \quad (19)$$

Example:

$$\begin{array}{l}
 \int_0^1 (x^2 + 2x) \cdot x \, dx \\
 \int_0^1 (x^3 + 2x^2) \, dx \\
 \int_0^1 (x^4 + 2x^3) \, dx
 \end{array}
 \left|
 \begin{array}{l}
 \text{Area} \\
 \text{1st moment} \\
 \text{2nd moment}
 \end{array}
 \right.
 \begin{array}{l}
 = \frac{1}{2} \\
 = \frac{1}{2} \\
 = \frac{1}{2}
 \end{array}
 \begin{array}{l}
 \text{cm} \\
 \text{cm} \\
 \text{cm}
 \end{array}$$

Example (2) is a particular case applied in the calculation of the first moment. The equations in a general way the next equations are used for the principle of Virtual Displacement Theory or the Principle of Virtual Work.

The equations (2) for the moment are:

$$\frac{1}{2} \int_0^1 (x^2 + 2x) \cdot x \, dx = \frac{1}{2} \int_0^1 (x^3 + 2x^2) \, dx$$

Example


Step 1:

$$\begin{array}{l}
 \frac{1}{2} \int_0^L (x^2 + 2x) \cdot x \, dx \\
 \frac{1}{2} \int_0^L (x^3 + 2x^2) \, dx
 \end{array}
 \left|
 \begin{array}{l}
 \text{Area} \\
 \text{1st moment}
 \end{array}
 \right.
 \begin{array}{l}
 = \frac{1}{2} \\
 = \frac{1}{2}
 \end{array}
 \begin{array}{l}
 \text{cm} \\
 \text{cm}
 \end{array}$$

Total Area under (2) is:

$$\frac{1}{2} \int_0^L (x^3 + 2x^2) \, dx$$

Step 2:

$$\begin{array}{l}
 \frac{1}{2} \int_0^L (x^3 + 2x^2) \, dx \\
 \frac{1}{2} \int_0^L (x^4 + 2x^3) \, dx
 \end{array}
 \left|
 \begin{array}{l}
 \text{Area} \\
 \text{1st moment}
 \end{array}
 \right.
 \begin{array}{l}
 = \frac{1}{2} \\
 = \frac{1}{2}
 \end{array}
 \begin{array}{l}
 \text{cm} \\
 \text{cm}
 \end{array}$$

Substituting w into the weak form and integrating by parts yields

$$\int_0^L \left(\frac{dw}{dx} \right)^2 dx = \int_0^L \left(\frac{dw}{dx} \right)^2 dx$$

This is the weak form of the problem.

The bilinear form is

$$a(w, v) = \int_0^L \frac{dw}{dx} \frac{dv}{dx} dx$$

and the linear form is

$$l(v) = \int_0^L f v dx$$

The Galerkin approximation is

$$\int_0^L \left(\frac{dw}{dx} \right)^2 dx = \int_0^L \frac{dw}{dx} \frac{dv}{dx} dx$$

The Galerkin approximation is

$$\int_0^L \left(\frac{dw}{dx} \right)^2 dx = \int_0^L \frac{dw}{dx} \frac{dv}{dx} dx$$

The Galerkin approximation is

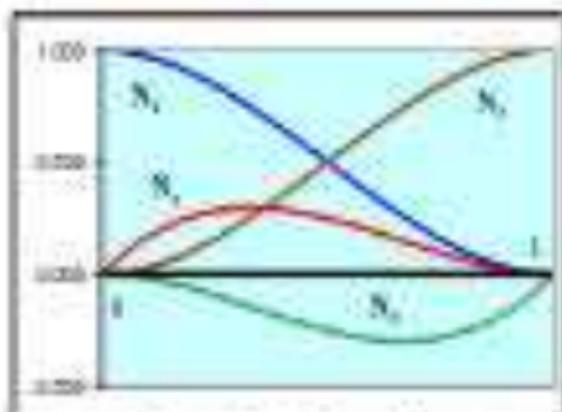
$$\int_0^L \left(\frac{dw}{dx} \right)^2 dx = \int_0^L \frac{dw}{dx} \frac{dv}{dx} dx$$

Beam Element

Simple Beam Element


$$\begin{bmatrix}
 \frac{EA}{L} & 0 \\
 0 & \frac{12EI}{L^3} \\
 0 & \frac{6EI}{L^2} \\
 0 & -\frac{6EI}{L^2} \\
 0 & 0 \\
 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2 \\
 v_2 \\
 \theta_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2 \\
 F_1 \\
 M_2
 \end{bmatrix}$$

- L Length
- E Young's Modulus of the beam material
- I Area Moment of Inertia
- $\frac{EA}{L}$ Axial Stiffness of Element
- $\frac{12EI}{L^3}$ Transverse Stiffness
- $\frac{6EI}{L^2}$ Shear Stiffness
- $-\frac{6EI}{L^2}$ Rotational Stiffness


Shape Functions for Beam Element

- assemble the global stiffness matrix
 - assemble the global force vector
 - solve the system of equations
 - extract the nodal values
 - post-process the results

$$\mathbf{K} \mathbf{u} = \mathbf{F}$$

where \mathbf{K} is the global stiffness matrix, \mathbf{u} is the global displacement vector, and \mathbf{F} is the global force vector.

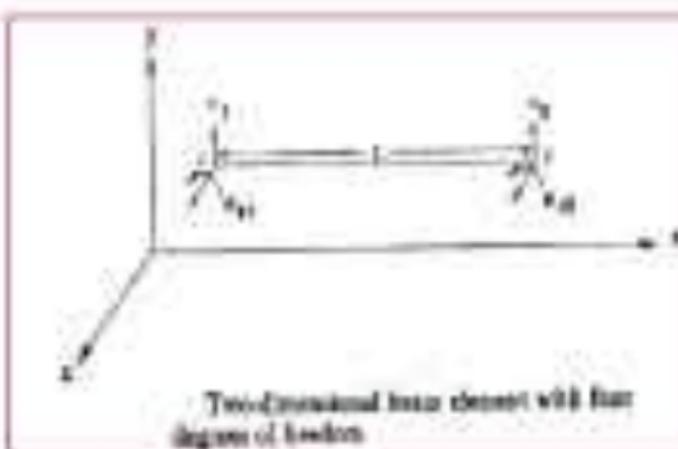
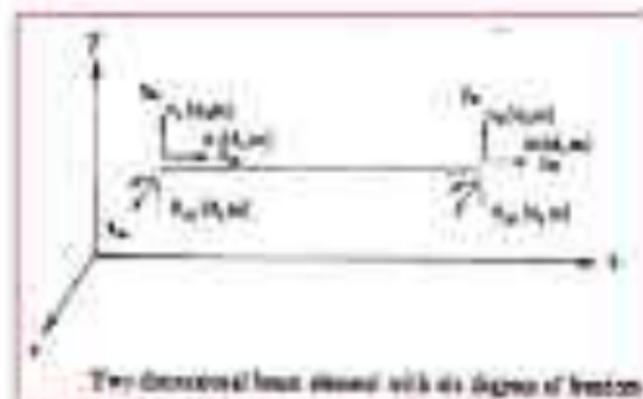
Node	1	2	3	4
u_1	12	18	0	0
u_2	18	36	18	0
u_3	0	18	36	18
u_4	0	0	18	36

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$$

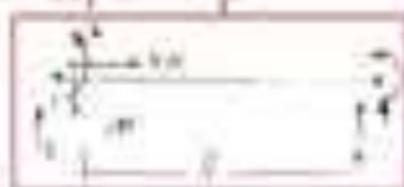
- assemble the global stiffness matrix
 - assemble the global force vector
 - solve the system of equations

$$\mathbf{K} \mathbf{u} = \mathbf{F}$$

Node	1	2	3	4
u_1	12	18	0	0
u_2	18	36	18	0
u_3	0	18	36	18
u_4	0	0	18	36

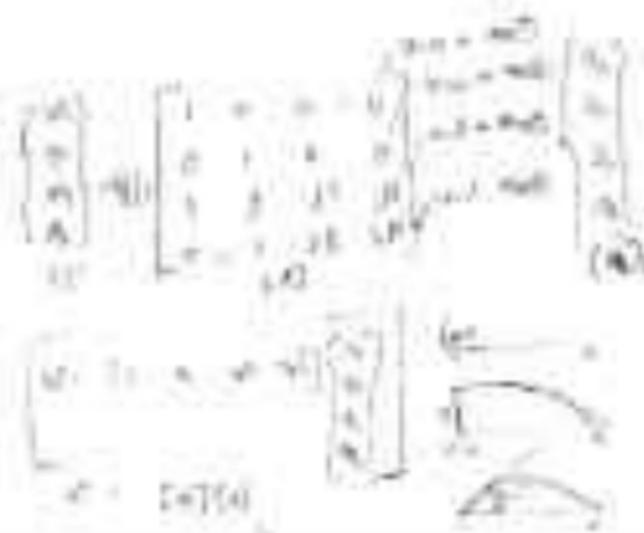
Two-Dimensional Beam Element


Development of shape function N for a bending element

 case of cube of end P
 V slope and θ slope


$$v = \frac{1}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L^2 & 0 \\ 0 & 0 & 0 & L^3 \end{bmatrix} \begin{bmatrix} v_P \\ \theta_P \\ V_P \\ F_P \end{bmatrix}$$

$$w = \frac{1}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & L^2 & 0 \\ 0 & 0 & 0 & L^3 \end{bmatrix} \begin{bmatrix} w_Q \\ \phi_Q \\ F_Q \\ V_Q \end{bmatrix}$$

 (9) $[N] \{S\}$


$$k_{11} = \int_0^L \left(-\frac{2}{L} + \frac{3x}{L^2} \right) \left(-\frac{2}{L} + \frac{3x}{L^2} \right) dx$$

$$= \frac{8L}{15}$$

$$k_{22} = \int_0^L \left(\frac{2x}{L} - \frac{x^2}{L^2} \right) \left(\frac{2x}{L} - \frac{x^2}{L^2} \right) dx$$

$$= \frac{8L}{15}$$

$$k_{12} = k_{21} = \int_0^L \left(-\frac{2}{L} + \frac{3x}{L^2} \right) \left(\frac{2x}{L} - \frac{x^2}{L^2} \right) dx$$

$$= -\frac{8L}{30}$$

$$k_{33} = \int_0^L \left(\frac{x}{L} \right) \left(\frac{x}{L} \right) dx$$

$$= \frac{L}{3}$$

$$k_{31} = k_{13} = \int_0^L \left(\frac{x}{L} \right) \left(-\frac{2}{L} + \frac{3x}{L^2} \right) dx$$

$$= -\frac{L}{6}$$

$$k_{32} = k_{23} = \int_0^L \left(\frac{x}{L} \right) \left(\frac{2x}{L} - \frac{x^2}{L^2} \right) dx$$

$$= \frac{L}{6}$$

$$k_{44} = \int_0^L \left(\frac{x}{L} \right) \left(\frac{x}{L} \right) dx$$

$$= \frac{L}{3}$$

$$k_{41} = k_{14} = \int_0^L \left(\frac{x}{L} \right) \left(-\frac{2}{L} + \frac{3x}{L^2} \right) dx$$

$$= -\frac{L}{6}$$

$$k_{42} = k_{24} = \int_0^L \left(\frac{x}{L} \right) \left(\frac{2x}{L} - \frac{x^2}{L^2} \right) dx$$

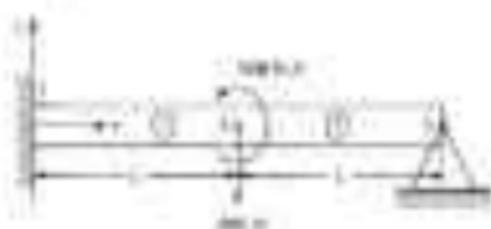
$$= \frac{L}{6}$$

$$k_{43} = k_{34} = \int_0^L \left(\frac{x}{L} \right) \left(\frac{x}{L} \right) dx$$

$$= \frac{L}{3}$$

$$k_{44} = \frac{8L}{15}$$

Beam Analysis : 1



Find the nodal loads & displacement at a fixed end & roller end

[Download presentation](#)

stiffness matrices

$$\begin{aligned}
 \mathbf{K}^{(1)} &= \frac{2E}{l} \begin{pmatrix} 11 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 11 \end{pmatrix} \\
 \mathbf{K}^{(2)} &= \frac{2E}{l} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 11 \end{pmatrix}
 \end{aligned}$$

4

The total stiffness matrix

$$\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \\ \mathbf{K}_3 \\ \mathbf{K}_4 \\ \mathbf{K}_5 \end{pmatrix} = \frac{2E}{l} \begin{pmatrix} 11 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11 \end{pmatrix}$$

boundary conditions

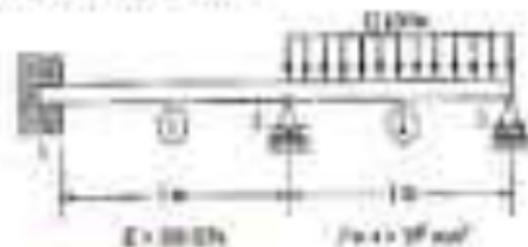
$$u_1 = 0 \quad u_2 = 0 \quad u_3 = 0$$

4

$$\begin{Bmatrix} -1000 \\ 1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^3 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} r_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

where $P_1 = -1000$ N, $Q_1 = 1000$ N, and $H_1 = 0$

determine (1) the slope at 2 and 3 and
 vertical deflection at the midpoint
 of the distributed load



Beam Analysis : 2



$$\frac{d^2}{dx^2} \leftarrow \frac{(200 \times 10^9)(4 \times 10^{-3})}{l^3} = 8 \times 10^7 \text{ N/m}$$

$$\mathbf{k} = \mathbf{k}^1 = 8 \times 10^7 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$\mathbf{u} = 10^3 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ -1000 \end{Bmatrix}$$

For element 2, $\varphi = 0$, $\psi = Q_3$, $\varphi = 2$ and $Q_4 = Q_5$.

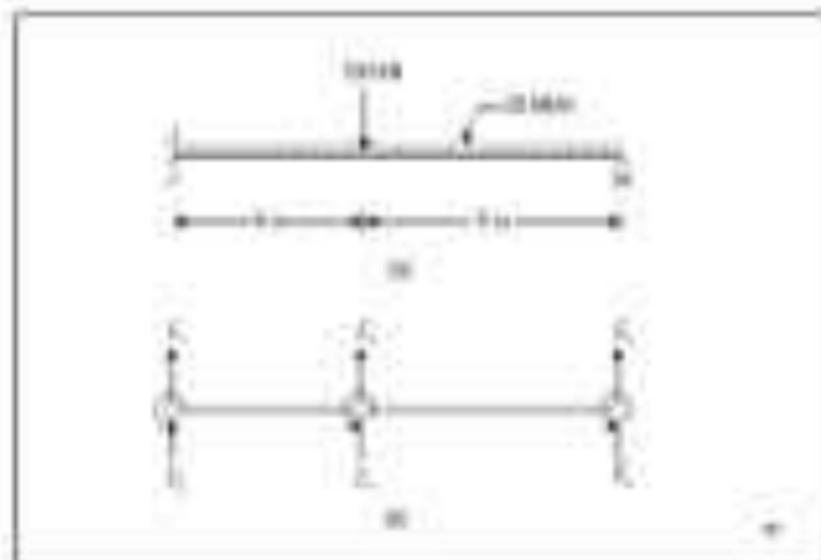
To get vertical deflection at the mid-point of the element, we

$$\begin{aligned} \psi &= \frac{1}{2}(1-\xi) = \frac{1}{2}(1-\xi) \\ \psi_{1/2} &= -\frac{\xi^2}{2} + \frac{\xi^3}{2} \\ \psi_{1/2} &= -\frac{1}{2} + \frac{1}{2} = 0 \\ \psi_{3/4} &= \frac{\xi^2}{2} - \frac{\xi^3}{2} \\ \psi_{3/4} &= -\frac{1}{2} + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \psi_1 &= 0.5, \psi_2 = 1.5 \\ \psi_3 &= 0.5, \psi_4 = -0.5 \end{aligned}$$

$$\mathbf{W} = \psi_1 \mathbf{u}_1 + \psi_2 \mathbf{u}_2$$

$$\begin{aligned} \mathbf{w} &= -833 \times 10^{-3} \text{ m} \\ &= -0.833 \text{ mm} \end{aligned}$$



$$[F] = \begin{pmatrix} -10 \\ -2000 \\ 10 \\ -1000 \\ -10 \\ 10 \end{pmatrix} \text{ Answer}$$

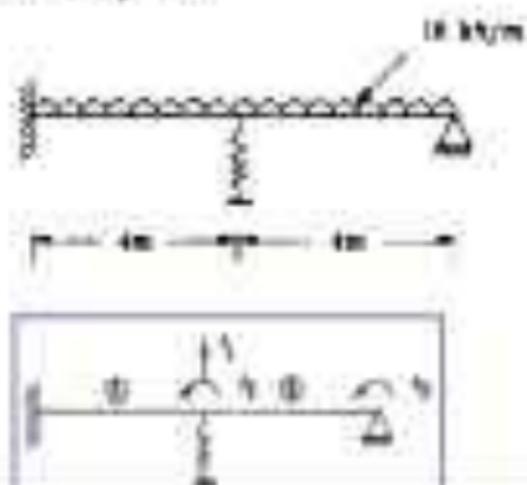
Answer

$$U_1 = -0.0133 \text{ m}$$

$$T_2 = -0.003 \text{ rad}$$

$$T_3 = 0.003 \text{ rad}$$

Beam Analysis : 3



$$[K] = \begin{matrix} \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \text{spr.} \end{matrix} \begin{bmatrix} 75 & 180 & -75 & 180 \\ 0 & 400 & -150 & 200 \\ 75 & -150 & 75 & -180 \\ 0 & 200 & -180 & 400 \end{bmatrix}$$

$$[K] = \begin{matrix} \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \text{spr.} \end{matrix} \begin{bmatrix} 75 & 180 & -75 & 180 \\ 0 & 400 & -150 & 200 \\ 75 & -150 & 75 & -180 \\ 0 & 200 & -180 & 400 \end{bmatrix}$$

$$\{U\} = -[K]^{-1}\{F\} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -20 \\ 20 \end{pmatrix}$$

$$\{U\} = -[K]^{-1}\{F\} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -20 \\ -20 \\ -20 \\ 20 \end{pmatrix}$$

Displacement of the wall support for stiffness $k = 100 \text{ kN/m}$
 All the members are fixed to the wall. Thus, the two fixed
 nodes at the wall is its displacement degrees of freedom r_1

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -20 \\ 0 \\ 20 \end{pmatrix}$$

Solving for three equations we get,

$$R_1 = -2000 = -0.200 \text{ MN}$$

$$R_2 = 0 = 0.000 \text{ MN}$$

$$R_3 = 2000 = 0.200 \text{ MN}$$

The load is represented by two member set (subsets) using

$$\{f\}_2 = K_1 \{u\}_1 + \{f\}_2$$

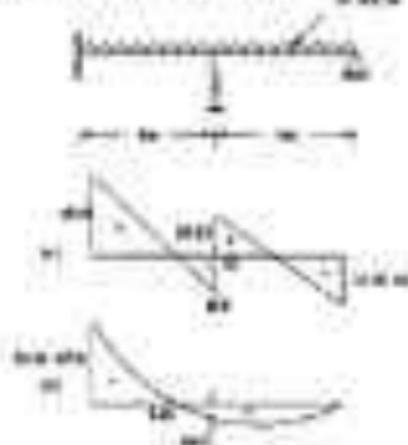
Matrix 1

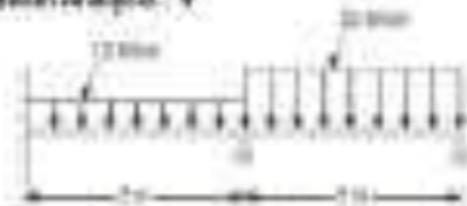
$$\begin{bmatrix} 75 & 150 & -15 & 15 \\ 150 & 300 & 15 & 15 \\ 15 & 15 & 100 & 0 \\ 15 & 15 & 0 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -8000 \\ -12000 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -20 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Matrix 2

$$\{f\}_2 = \begin{bmatrix} 15 & 30 & -15 & 15 \\ 30 & 60 & 15 & 15 \\ 15 & 15 & 100 & 0 \\ 15 & 15 & 0 & 100 \end{bmatrix} \begin{Bmatrix} -0.01 \\ -0.01 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -20 \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ -0.04 \\ 0.00 \\ 0.00 \end{Bmatrix}$$

Using the above values of fixed end forces on the members 1 and 2, the shear force and bending moment diagrams are drawn as shown in figure.



Beam Analysis: 4


$$E = 2 \times 10^8 \text{ kN/m}^2 \text{ and } I = 1 \times 10^4 \text{ m}^4$$



$$\begin{aligned}
 u_1 &= 2 \times 10^8 \text{ kN/m}^2 \times 1 \times 10^4 \text{ m}^4 = 2 \times 10^{12} \text{ kN} \\
 u_2 &= 1 \times 10^8 \text{ kN/m}^2 \times 1 \times 10^4 \text{ m}^4 \\
 u_3 &= 2 \times 10^8 \text{ kN/m}^2 \times 1 \times 10^4 \text{ m}^4
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{EI} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} & \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{EI} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} & \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
 \end{aligned}$$

Ans)

$$\frac{1}{E} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Elemental stiffness is given by

$$\frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
 \vec{d}^T &= \begin{bmatrix} (20) \times 1 \\ 0 \\ (20) \times 1 \\ 12 \\ (20) \times 1 \\ 0 \\ (4 \cdot 20) \times 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 20 \\ 12 \\ 20 \\ 0 \\ 80 \\ 12 \end{bmatrix} \\
 &= \begin{bmatrix} 20 \\ 0 \\ 20 \\ 12 \\ 20 \\ 0 \\ 80 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 20 \\ 12 \\ 20 \\ 0 \\ 80 \\ 12 \end{bmatrix}
 \end{aligned}$$

3. The stiffness system is

$$\mathbf{K} \cdot \vec{d} = \vec{d}^T$$

$$\begin{bmatrix} 12 & 0 & -12 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 \\ -12 & 0 & 24 & 0 & -12 & 0 \\ 0 & 0 & 0 & 20 & -20 & 0 \\ 0 & 0 & -12 & -20 & 12 & -8 \\ 0 & 0 & 0 & 0 & -20 & 20 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 20 \\ 12 \\ 20 \\ 0 \end{bmatrix}$$

Boundary condition:

In the three points by means condition:

$$d_1 = d_2 = d_3 = d_4 = 0$$

$$6 \begin{bmatrix} 200 & 80 \\ 50 & 100 \end{bmatrix} \begin{bmatrix} \delta_4 \\ \delta_5 \end{bmatrix} = \begin{bmatrix} -25 \\ 50 \end{bmatrix}$$

$$600 \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \delta_4 \\ \delta_5 \end{bmatrix} = \begin{bmatrix} -25 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} \delta_4 \\ \delta_5 \end{bmatrix} = \frac{1}{600} \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -25 \\ 50 \end{bmatrix}$$

$$= \frac{1}{600} \frac{1}{4-4} \begin{bmatrix} 2 & -4 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -25 \\ 50 \end{bmatrix} = \frac{1}{2000} \begin{bmatrix} -100 \\ 225 \end{bmatrix} \quad \text{Answer}$$

End structure

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 20 & -12 & 20 \\ 20 & 100 & -20 & 20 \\ -12 & -20 & 12 & -20 \\ 20 & 20 & -20 & 100 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -100 \end{bmatrix}$$

$$\delta_1 = \delta_2 = \delta_3 = 0 \text{ and } \delta_4 = \frac{-100}{200} = \frac{-100}{200} \quad \text{Answer}$$

For element 2

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 22 & 10 & 42 & 10 \\ 10 & 100 & 40 & 10 \\ -11 & 40 & 22 & -10 \\ 10 & 40 & -40 & 100 \end{bmatrix} \begin{bmatrix} 0 \\ -100 \\ 200 \\ -200 \end{bmatrix} = \begin{bmatrix} -60 \\ 100 \\ 6720 \\ 0 \end{bmatrix} \quad \text{Answer}$$

Deflection at each node

$$u = [N_1 \ N_2 \ N_3 \ N_4] \{U\}_e$$

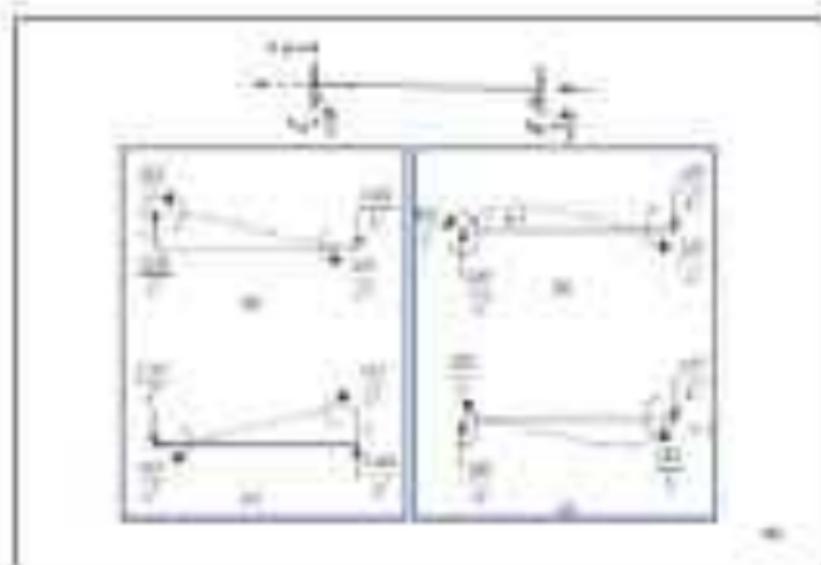
 For element 1, $u = 0.03332$ at $x = 22.52$ mm

 For element 2, $u = 27.9$ mm Aravamudan Answer

**Stiffness Matrix for a Two Dimensional Beam
(Element) with Six Degrees of Freedom**


Two dimensional beam element with six degrees of freedom

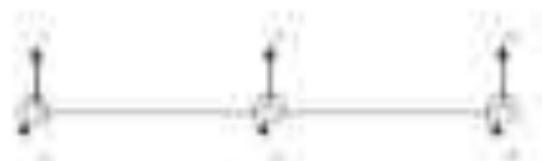
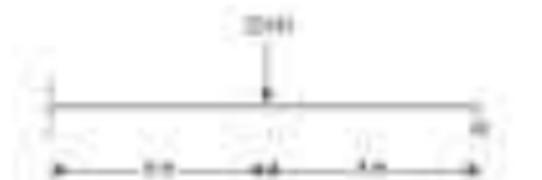
$$\begin{array}{c}
 \mathbf{u} \\
 \hline
 \begin{array}{ccccccc}
 u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\
 \hline
 u_1 & 5 & 0 & 0 & 0 & 0 & 0 \\
 u_2 & 0 & 5 & 0 & 0 & 0 & 0 \\
 u_3 & 0 & 0 & 5 & 0 & 0 & 0 \\
 u_4 & 0 & 0 & 0 & 5 & 0 & 0 \\
 u_5 & 0 & 0 & 0 & 0 & 5 & 0 \\
 u_6 & 0 & 0 & 0 & 0 & 0 & 5 \\
 u_7 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array}
 \end{array}$$



- definition:**
1. Deflection under load
 2. Shear force and bending moment at each span
 3. Reaction at supports

$$I = 50000 \text{ cm}^4 = 20 \times 10^8 \text{ cm}^4 = 20 \times 10^{-8} \text{ m}^4$$

constant of rigidity as $24 \times 10^8 \text{ cm}^2$



Node displacement vector is

$$\{d\}^T = \{d_1, d_2, d_3, d_4, d_5, d_6\}$$

Stiffness matrix for the element

$$k = \frac{EA}{L} \begin{bmatrix} 12 & 6e & -12 & 6e & 0 & 0 \\ 6e & 4e^2 & -6e & 2e^2 & 0 & 0 \\ -12 & -6e & 12 & -6e & 0 & 0 \\ 6e & -2e^2 & 6e & 4e^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow k = \frac{900}{127} \begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 \\ 60 & 180 & -60 & 180 & 0 & 0 \\ -12 & -60 & 12 & -60 & 0 & 0 \\ 60 & -180 & 60 & 180 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$k = \frac{900}{127} \begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 \\ 60 & 180 & -60 & 180 & 0 & 0 \\ -12 & -60 & 12 & -60 & 0 & 0 \\ 60 & -180 & 60 & 180 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Once stiffness matrix

$$\{F\} = \frac{900}{127} \begin{bmatrix} 12 & 60 & -12 & 60 & 0 & 0 \\ 60 & 180 & -60 & 180 & 0 & 0 \\ -12 & -60 & 12 & -60 & 0 & 0 \\ 60 & -180 & 60 & 180 & 0 & 0 \\ 0 & 0 & -12 & -60 & 12 & 60 \\ 0 & 0 & 0 & 0 & -60 & 180 \end{bmatrix} \{d\}$$

Discretized equilibrium equations:

$$[K] \{d\} = \{F\} - \{F_0\}$$

Stiffness equations:

$$[K] \{d\} = \{F\}$$

Nodal coordinates:

$$x_1, x_2, x_3, x_4$$

Applying fixed boundary conditions:

$$\frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 36 \\ 0 & 240 & 0 \\ 36 & 0 & 108 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \frac{L^3}{EI} \frac{1}{240000} \begin{bmatrix} 1700 & 0 & -600 \\ 0 & 160 & 0 \\ -600 & 0 & 300 \end{bmatrix} \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \frac{1}{EI} \begin{bmatrix} -101.33 \\ -11.43 \\ 0 \end{bmatrix} = \frac{1}{800} \begin{bmatrix} -101.33 \\ -11.43 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.127 \\ -0.0143 \\ 0 \end{bmatrix}$$

Shear Force and bending Moment at nodes

Considering element (1), for nodal eqns of forces

$$k_1 = \frac{11}{2} \times 10^6 \text{ N/m} \Rightarrow k_1 \frac{1}{10^6} \begin{Bmatrix} 0 \\ 0 \\ -412.101 \\ -11621 \end{Bmatrix}$$

$$= \frac{1}{10^6} \begin{Bmatrix} 0 & 0 & -4 & -2 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -412.101 \\ -11621 \end{Bmatrix} = 11.276 \text{ kN/m}$$

$$F = \frac{25}{120} \begin{Bmatrix} 12 & 6 & 6 & -12 \\ 6 & 6 & -6 & 6 \end{Bmatrix} \frac{1}{25} \begin{Bmatrix} 0 \\ 0 \\ -412.101 \\ -11621 \end{Bmatrix} = 16.714 \text{ kN}$$

Considering element 2, the nodal eqns are given by

End Displacements

At support on left end i.e. from element 1

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{25}{120} \begin{Bmatrix} 12 & 6 & 6 & -12 \\ 6 & 6 & -6 & 6 \end{Bmatrix} \frac{1}{25} \begin{Bmatrix} 0 \\ 0 \\ -412.101 \\ -11621 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

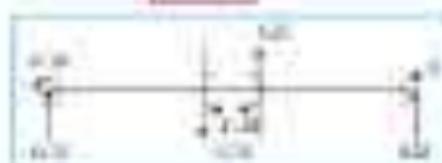
$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 16.714 \\ 17.146 \end{Bmatrix}$$

Considering element 2:

The stresses at node based elements can be derived:

$$\left\{ \begin{matrix} \sigma \\ \sigma_n \end{matrix} \right\} = \frac{E}{L} \begin{bmatrix} 32 & -42 & -24 & 32 & -36 \\ 12 & 18 & 18 & -36 & 36 \end{bmatrix} \frac{1}{21} \begin{bmatrix} -42720 \\ -42720 \\ 0 \\ 0 \\ 42.5 \end{bmatrix} = \begin{bmatrix} -160 \\ 160 \end{bmatrix}$$

$$= \begin{bmatrix} 627 \\ 0 \end{bmatrix}$$



Dean Example



- Task 1: The beam is fixed against rotation at the left and right ends and carries the load P as shown in the figure.
- Task 2: Find the local axial displacements and stresses within the beam as a function of the axial coordinate x .

Elemental stiffnesses are:

$$k_1 = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & -6 & 0 \\ 0 & -6 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \quad k_2 = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 12 & -6 & 0 \\ -6 & -6 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

Global Equations

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}
 \begin{Bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4 \\
 u_5 \\
 u_6
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{Bmatrix}$$

Each node receives 30 kN force

$$F_n = P = 30 \text{ kN}$$

$$u_1 = u_2 = u_3 = u_4 = u_5 = u_6$$

Reduced to single

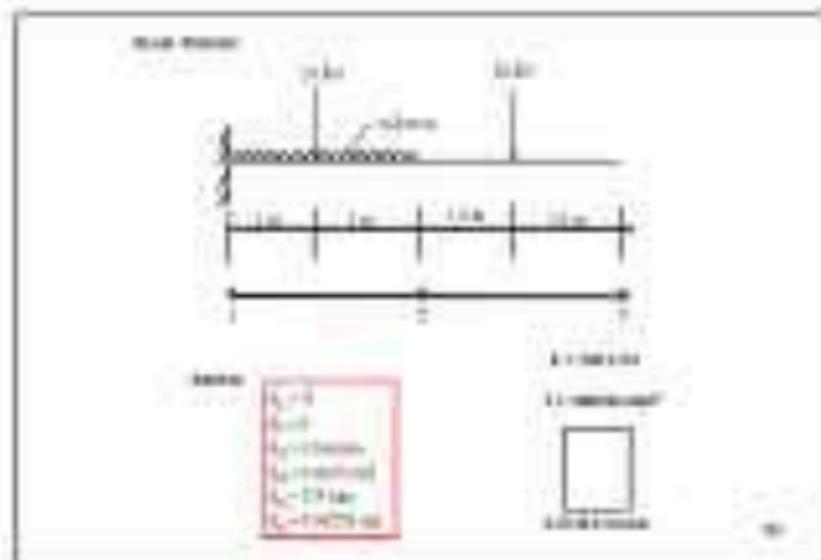
$$\begin{Bmatrix}
 6 \\
 6 \\
 6 \\
 6 \\
 6 \\
 6
 \end{Bmatrix}
 \begin{Bmatrix}
 u \\
 u \\
 u \\
 u \\
 u \\
 u
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 -P \\
 -P \\
 -P \\
 -P \\
 -P \\
 -P
 \end{Bmatrix}$$

 Solving for u gives

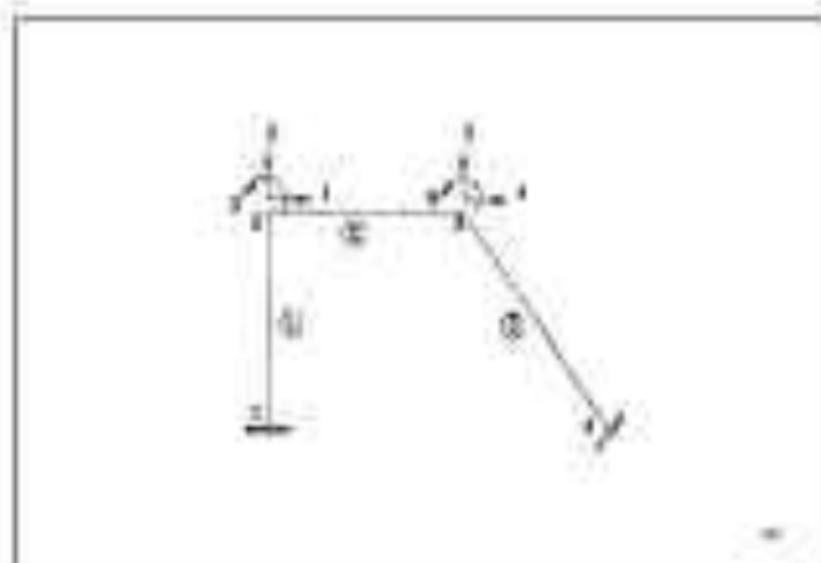
$$\begin{Bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4 \\
 u_5 \\
 u_6
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 -P/6 \\
 -P/6 \\
 -P/6 \\
 -P/6 \\
 -P/6 \\
 -P/6
 \end{Bmatrix}$$

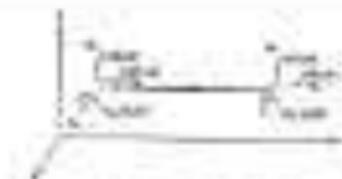
Total global force applied to the bar is twice that of each node

$$\begin{Bmatrix}
 F_1 \\
 F_2 \\
 F_3 \\
 F_4 \\
 F_5 \\
 F_6
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 -2P \\
 -2P \\
 -2P \\
 -2P \\
 -2P \\
 -2P
 \end{Bmatrix}
 \begin{Bmatrix}
 u \\
 u \\
 u \\
 u \\
 u \\
 u
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 -2P + 2P = 0 \\
 -2P + 2P = 0
 \end{Bmatrix}$$



2D Frame Analysis





Two-dimensional beam element with six degrees of freedom

$$\mathbf{K}_e = \begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & 0 & 0 & 0 & 0 \\
 0 & 0 & \frac{4EI}{L} & 0 & 0 & 0 \\
 0 & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
 0 & 0 & 0 & 0 & \frac{12EI}{L^3} & 0 \\
 0 & 0 & 0 & 0 & 0 & \frac{4EI}{L}
 \end{bmatrix}$$

Member 1

$$E = 4 \times 10^7 \text{ (N/m}^2\text{)}, \quad I = 10 \times 10^6 \text{ (m}^4\text{)}, \quad L = 0.02 \text{ (m)}$$

$$k_1 = 12 \times 10^9 \text{ (N/m}^2\text{)}, \quad A_1 = 0.0006 \text{ (m}^2\text{)}$$

$$\mathbf{K}_e = \begin{bmatrix}
 6.6 \times 10^9 & 0 & 0 & -0.1 \times 10^6 & 0 & 0 \\
 0 & 60 & 0 & 0 & -60 & 0 \\
 0 & 0 & 200 & 0 & -60 & 120 \\
 -0.1 \times 10^6 & 0 & 0 & 6.6 \times 10^9 & 0 & 0 \\
 0 & -60 & -60 & 0 & 60 & -60 \\
 0 & 0 & 120 & 0 & -60 & 60
 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix}
 2E & 0 & 0 & 0 & 0 & 0 \\
 0 & 2E & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix} \quad C_p = 0, C_r = 1.4$$

$$\mathbf{K} = \begin{bmatrix}
 2 & 0 & 0 & 0 & 0 & 0 \\
 0 & 2 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}$$

The stiffness matrix \mathbf{K} is global and given by:

$$\mathbf{K}_G = \mathbf{17}^T \mathbf{K} \mathbf{17}$$

$$\mathbf{K}_G = \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}$$

Member 2

Same formulation is related to the global direction, no transformation is required. The member stiffness matrix in the local direction is the same as in the case of member 1.

$$[k] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 \times 10^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix}$$

The fixed end values also do not require any transformation.

$$\{Q_2\} = \begin{Bmatrix} 0 \\ 0 \\ \frac{1000}{2} \\ \frac{1000}{2} \\ 0 \\ 1000 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ -100 \\ 0 \\ -100 \\ 100 \end{Bmatrix}$$

Member 3

$$k_{el} = \begin{bmatrix} 11 \times 10^7 & 0 & 0 & -11 \times 10^7 & 0 & 0 \\ 0 & 22 & 10 & 0 & -22 & 10 \\ 0 & 12 & 100 & 0 & -12 & 100 \\ -11 \times 10^7 & 0 & 0 & 11 \times 10^7 & 0 & 0 \\ 0 & -22 & -10 & 0 & 22 & -10 \\ 0 & 12 & -100 & 0 & -12 & 100 \end{bmatrix}$$

$$C_1 = 0.5, C_2 = -0.5$$

$$[T] = \begin{bmatrix} 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$[k] = [T]^T k_{el} [T]$$

$$[k] = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix}
 10000 & 0 & 0 & -10000 & 0 & 0 \\
 0 & 20000 & 0 & 0 & -100 & 0 \\
 0 & 0 & 400 & 0 & -100 & 100 \\
 -10000 & 0 & 0 & 10000 & -1000 & 100 \\
 0 & -100 & -100 & -1000 & 1010 & -100 \\
 0 & 0 & 100 & 100 & -100 & 100
 \end{bmatrix}
 \begin{Bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4 \\
 u_5 \\
 u_6
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{Bmatrix}$$

$$\mathbf{u} = \begin{Bmatrix}
 0.000 \\
 -0.001 \\
 -0.002 \\
 0.0 \\
 -0.001 \\
 0.003
 \end{Bmatrix}$$

$$\mathbf{p} = \begin{Bmatrix}
 0.2008 \\
 0.0016073 \\
 -0.0008 \\
 0.2010 \\
 0.2007 \\
 0.0000
 \end{Bmatrix}$$

41

Abstrakt Ein-Feld

where

$$\{N\} = [k_u] \{d_u\} + \{f_u\}$$

$$[k_u] = [11 \ 14]$$

Member 1

$$[k_u] = \begin{bmatrix}
 0 & 1 & 0 & 0 & 0 & 0 \\
 -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}
 \begin{Bmatrix}
 0 \\
 0 \\
 0.0001 \\
 0.0011 \cdot 10^{-4} \\
 -0.0001 \\
 0
 \end{Bmatrix}$$

$$= \begin{Bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0.0001 \cdot 10^{-4} \\
 -0.0001 \\
 -0.0001
 \end{Bmatrix}$$

42

$$\begin{aligned}
 & \mathbf{U}^T = \mathbf{K}_1 \mathbf{U}_1 + \mathbf{U}_2 \\
 & \mathbf{U} = \begin{pmatrix} 15 \times 10^9 & 0 & 0 & -2 \times 10^9 & 0 & 0 \\ 0 & 80 & 0 & 0 & -40 & 0 \\ 0 & 0 & 200 & 0 & -20 & 20 \\ -2 \times 10^9 & 0 & 0 & 15 \times 10^9 & 0 & 0 \\ 0 & -40 & -20 & 0 & 80 & -40 \\ 0 & 0 & 20 & 0 & -20 & 200 \end{pmatrix} \\
 & \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1.072 \times 10^{-4} \\ -6.250 \\ -6.000 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -14.11 \\ 57.57 \\ 21.81 \\ 13.12 \\ -22.87 \\ 161.84 \end{pmatrix}
 \end{aligned}$$

Master 2

Use the master nodes per panel to the global nodes $\mathbf{U}_2 = \mathbf{U}$

$$\begin{aligned}
 & \mathbf{U} = \begin{pmatrix} 15 \times 10^9 & 0 & 0 & -2 \times 10^9 & 0 & 0 \\ 0 & 80 & 0 & 0 & -40 & 0 \\ 0 & 0 & 200 & 0 & -20 & 20 \\ -2 \times 10^9 & 0 & 0 & 15 \times 10^9 & 0 & 0 \\ 0 & -40 & -20 & 0 & 80 & -40 \\ 0 & 0 & 20 & 0 & -20 & 200 \end{pmatrix} \\
 & \begin{pmatrix} -1.509 \\ 1.050 \times 10^{-4} \\ -4.296 \\ 1.206 \\ 5.239 \\ -2.760 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.0 \\ -0.0 \end{pmatrix} = \begin{pmatrix} 0.0 \\ -0.0 \\ -0.0 \\ 0.0 \\ -0.0 \\ -0.0 \end{pmatrix}
 \end{aligned}$$

Master 2

$$[k_u] = [T]^T [k]$$

$$[k_u] = \begin{bmatrix} 75.0 & -0.0 & 0 & 0 & 0 & 0 \\ 0.0 & 60.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & -0.0 & 0 \\ 0 & 0 & 0 & 0.0 & 0.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1.000 \\ 0.000 \\ 0.000 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} 75.0 \cdot 10^{-6} \\ 0.000 \\ 0.000 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$$

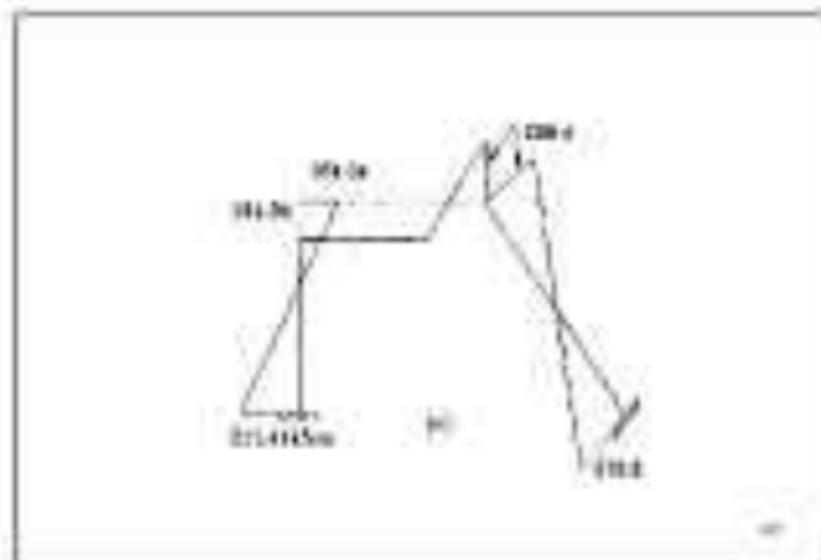
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$$[F] = [k_u] [u_d] + [F_0]$$

$$[F] = \begin{bmatrix} 75 \cdot 10^6 & 0 & 0 & -10 \cdot 10^6 & 0 & 0 \\ 0 & 200 & 700 & 0 & -100 & 700 \\ 0 & 0 & 700 & 0 & -700 & 700 \\ -10 \cdot 10^6 & 0 & 0 & 10 \cdot 10^6 & 0 & 0 \\ 0 & -100 & -700 & 0 & 200 & -100 \\ 0 & 700 & 700 & 0 & -700 & 1000 \end{bmatrix} \cdot$$

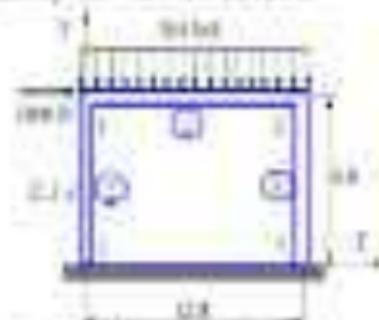
$$\begin{bmatrix} 2.5 \cdot 10^{-4} \\ 0.0040 \\ 0.0100 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} -200.0 \\ -114.4 \\ 200.1 \\ -502.0 \\ -114.4 \\ 271.8 \end{bmatrix}$$

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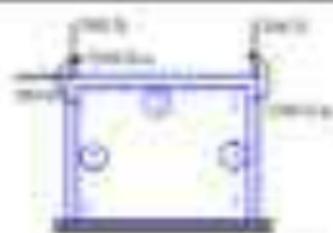
4.2 Analysis of Beam Structure

Members in this structure are subjected to uniformly distributed load (UDL) and moments. It is very easy to find out the reactions. It is possible to find the stress distribution of the members from classical mechanics of beam.



Given: UDL = 100 kN/m, $10 \times 10 \text{ m}^2$ structure

Find: Reactions and stress distribution of the members.



The corresponding nodal displacement degrees of freedom are:

Node	1	2	3	4
1	1	2	3	4
2	5	6	7	8
3	9	10	11	12
4	13	14	15	16

Element stiffness table

Element	Node 1	Node 2	Node 3	Node 4
1	1	2	3	4
2	5	6	7	8
3	9	10	11	12
4	13	14	15	16

Element Stiffness

Element	1	2	3	4	5	6
1	100	0	0	0	0	0
2	0	100	0	0	0	0
3	0	0	100	0	0	0
4	0	0	0	100	0	0
5	0	0	0	0	100	0
6	0	0	0	0	0	100

Einsetzen in (4) und Auflösen nach u
 (mit \mathbf{K}^{-1})

$$\mathbf{K}^{-1} \cdot \mathbf{K} \cdot \mathbf{u} = \mathbf{K}^{-1} \cdot \mathbf{F}$$

$$\mathbf{u} = \mathbf{K}^{-1} \cdot \mathbf{F}$$

Einsetzen in (3)

$$\mathbf{K} \cdot \mathbf{K}^{-1} \cdot \mathbf{F} = \mathbf{F}$$

(4)

$$\mathbf{u} = \mathbf{K}^{-1} \cdot \mathbf{F}$$

(5)

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matrix $\mathbf{K}^{(1)}$ from assembly
 iteration:

$$\begin{aligned}
 & \mathbf{K}^{(1)} = \mathbf{K}^{(0)} + \mathbf{K}^{(1)} \\
 & \mathbf{K} = 100 \cdot \mathbf{K}^{(1)} \quad \mathbf{K} = \mathbf{K}^{(1)} + \mathbf{K}^{(2)} \\
 & \mathbf{K} = 100 \cdot \mathbf{K}^{(1)} + 100 \cdot \mathbf{K}^{(2)}
 \end{aligned}$$

matrix $\mathbf{K}^{(1)}$ from

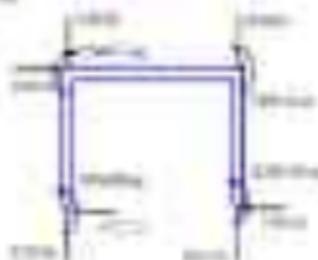
$$\mathbf{K}^{(1)} = \begin{pmatrix}
 100 & 0 & 0 & 0 & 0 & 0 \\
 0 & 100 & 0 & 0 & 0 & 0 \\
 0 & 0 & 100 & 0 & 0 & 0 \\
 0 & 0 & 0 & 100 & 0 & 0 \\
 0 & 0 & 0 & 0 & 100 & 0 \\
 0 & 0 & 0 & 0 & 0 & 100
 \end{pmatrix}$$

K = I

$$\mathbf{K} = \begin{pmatrix}
 100 & 0 & 0 & 0 & 0 & 0 \\
 0 & 100 & 0 & 0 & 0 & 0 \\
 0 & 0 & 100 & 0 & 0 & 0 \\
 0 & 0 & 0 & 100 & 0 & 0 \\
 0 & 0 & 0 & 0 & 100 & 0 \\
 0 & 0 & 0 & 0 & 0 & 100
 \end{pmatrix}$$

Solving $\mathbf{K} \cdot \mathbf{u} = \mathbf{f}$

$$\begin{aligned}
 u_1 &= 0.01 \text{ m} \\
 u_2 &= 0.01 \text{ m} \\
 u_3 &= 0.01 \text{ m} \\
 u_4 &= 0.01 \text{ m} \\
 u_5 &= -0.01 \text{ m} \\
 u_6 &= 0.01 \text{ m}
 \end{aligned}$$



Comparison of Element Load Vectors

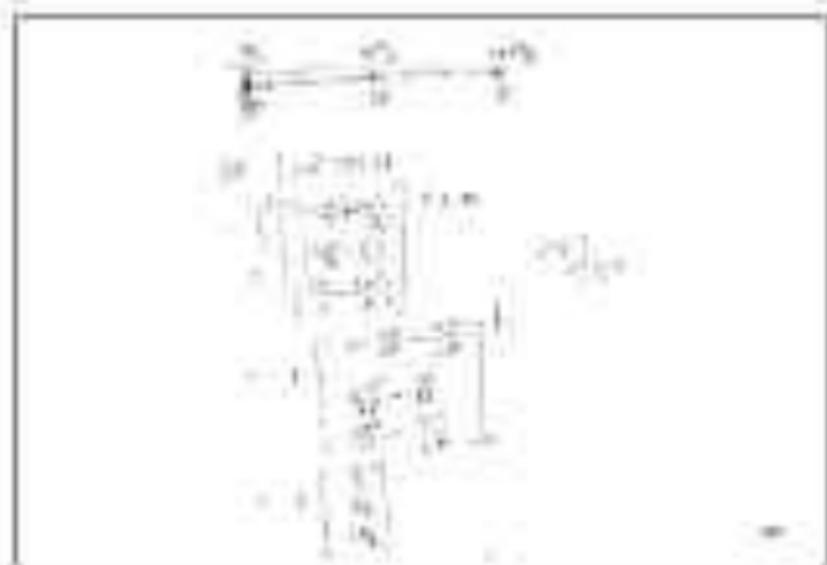
Consider a beam element of length L and cross-sectional area A . The element is subjected to a uniformly distributed load w acting downwards. The nodal load vectors are compared as follows:

$$\mathbf{F}_1 = \int_0^L \mathbf{f}^T dx = \int_0^L \begin{bmatrix} 0 \\ -w \\ 0 \end{bmatrix} dx = \begin{bmatrix} 0 \\ -wL \\ 0 \end{bmatrix}$$

The nodal load vector at node 1 is \mathbf{F}_1 . The nodal load vector at node 2 is \mathbf{F}_2 . The total nodal load vector is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$.

$$\mathbf{F} = \begin{bmatrix} 0 \\ -wL \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -wL \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2wL \\ 0 \end{bmatrix}$$

The nodal load vector at node 1 is $\mathbf{F}_1 = \begin{bmatrix} 0 \\ -wL \\ 0 \end{bmatrix}$. The nodal load vector at node 2 is $\mathbf{F}_2 = \begin{bmatrix} 0 \\ -wL \\ 0 \end{bmatrix}$.

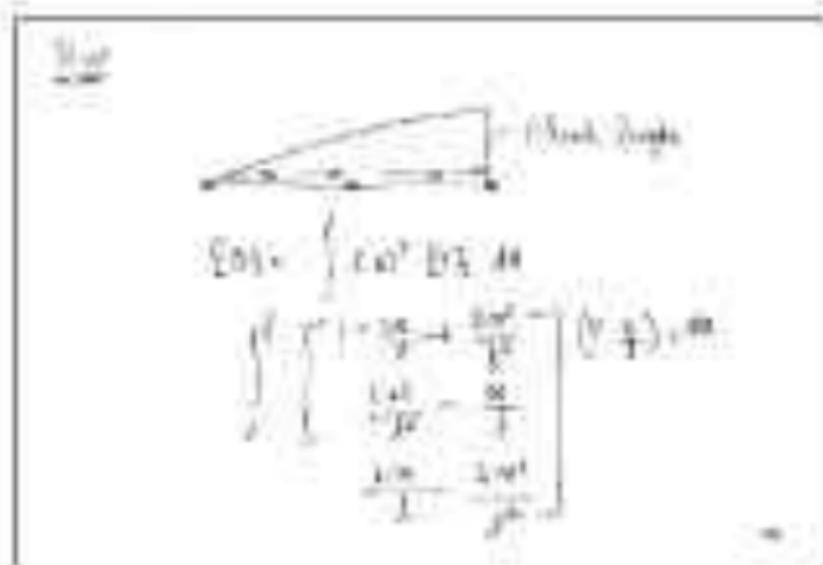
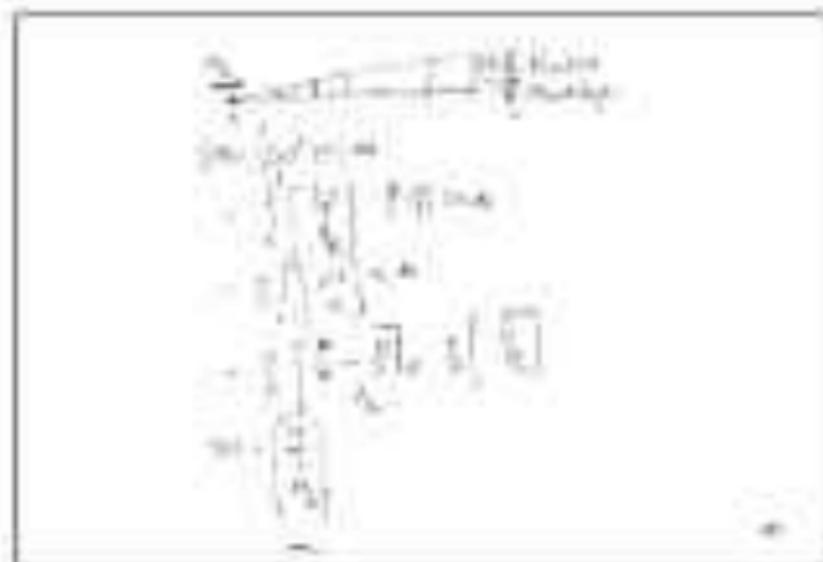


Ex: $\int_{-1}^1 (1-x^2) dx = \frac{4}{3}$

$$\begin{aligned}
 & \int_{-1}^1 (1-x^2) dx = \int_{-1}^1 \left[\frac{1}{2} \left(\frac{1-x}{2} \right) + \frac{1}{2} \left(\frac{1+x}{2} \right) \right] dx \\
 & = \int_{-1}^1 \left[\frac{1-x}{2} \right] dx + \int_{-1}^1 \left[\frac{1+x}{2} \right] dx \\
 & = \left[\frac{x}{2} - \frac{x^2}{4} \right]_{-1}^1 + \left[\frac{x}{2} + \frac{x^2}{4} \right]_{-1}^1 \\
 & = \left[\frac{1}{2} - \frac{1}{4} \right] - \left[-\frac{1}{2} - \frac{1}{4} \right] + \left[\frac{1}{2} + \frac{1}{4} \right] - \left[-\frac{1}{2} + \frac{1}{4} \right] \\
 & = \left[\frac{1}{4} + \frac{1}{4} \right] + \left[\frac{1}{4} + \frac{1}{4} \right] = \frac{1}{2} + \frac{1}{2} = 1
 \end{aligned}$$

Ex: $\int_{-1}^1 (1-x^2) dx = \frac{4}{3}$

$$\begin{aligned}
 & \int_{-1}^1 (1-x^2) dx = \int_{-1}^1 \left[\frac{1}{2} \left(\frac{1-x}{2} \right) + \frac{1}{2} \left(\frac{1+x}{2} \right) \right] dx \\
 & = \int_{-1}^1 \left[\frac{1-x}{2} \right] dx + \int_{-1}^1 \left[\frac{1+x}{2} \right] dx \\
 & = \left[\frac{x}{2} - \frac{x^2}{4} \right]_{-1}^1 + \left[\frac{x}{2} + \frac{x^2}{4} \right]_{-1}^1 \\
 & = \left[\frac{1}{2} - \frac{1}{4} \right] - \left[-\frac{1}{2} - \frac{1}{4} \right] + \left[\frac{1}{2} + \frac{1}{4} \right] - \left[-\frac{1}{2} + \frac{1}{4} \right] \\
 & = \left[\frac{1}{4} + \frac{1}{4} \right] + \left[\frac{1}{4} + \frac{1}{4} \right] = \frac{1}{2} + \frac{1}{2} = 1
 \end{aligned}$$



The diagram shows a beam of length L with a pin support at the left end, a roller support at the midpoint, and a fixed support at the right end. A uniformly distributed load w is applied downwards over the entire length of the beam.

The beam is divided into two segments of length $L/2$ each. The left segment is labeled "Segment 1" and the right segment is labeled "Segment 2".

The nodal degrees of freedom (DOFs) are defined as follows:

- Node 1 (Left end): u_1 (horizontal displacement), v_1 (vertical displacement), θ_1 (rotation)
- Node 2 (Midpoint): u_2 (horizontal displacement), v_2 (vertical displacement), θ_2 (rotation)
- Node 3 (Right end): u_3 (horizontal displacement), v_3 (vertical displacement), θ_3 (rotation)

The global stiffness matrix K is assembled as follows:

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

The global load vector F is assembled as follows:

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

The global displacement vector U is assembled as follows:

$$U = \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}$$

The global equilibrium equation is:

$$K U = F$$

The beam is divided into two segments of length $L/2$ each. The left segment is labeled "Segment 1" and the right segment is labeled "Segment 2".

The nodal degrees of freedom (DOFs) are defined as follows:

- Node 1 (Left end): u_1 (horizontal displacement), v_1 (vertical displacement), θ_1 (rotation)
- Node 2 (Midpoint): u_2 (horizontal displacement), v_2 (vertical displacement), θ_2 (rotation)
- Node 3 (Right end): u_3 (horizontal displacement), v_3 (vertical displacement), θ_3 (rotation)

The global stiffness matrix K is assembled as follows:

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

The global load vector F is assembled as follows:

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

The global displacement vector U is assembled as follows:

$$U = \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}$$

The global equilibrium equation is:

$$K U = F$$

Super Node for primary supports of a frame
 that is fixed at certain points.

$$\int_V \sigma \cdot \delta u = \frac{P \Delta}{(2 \times 10^8)}$$

$$\int_V \sigma \cdot \delta u = \frac{P \Delta}{(2 \times 10^8)}$$

$$\int_V \sigma \cdot \delta u = \frac{P \Delta}{(2 \times 10^8)}$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

$$W = \frac{1}{2} P \Delta = 1000000$$

$$W = \frac{1}{2} P \Delta = 1000000$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

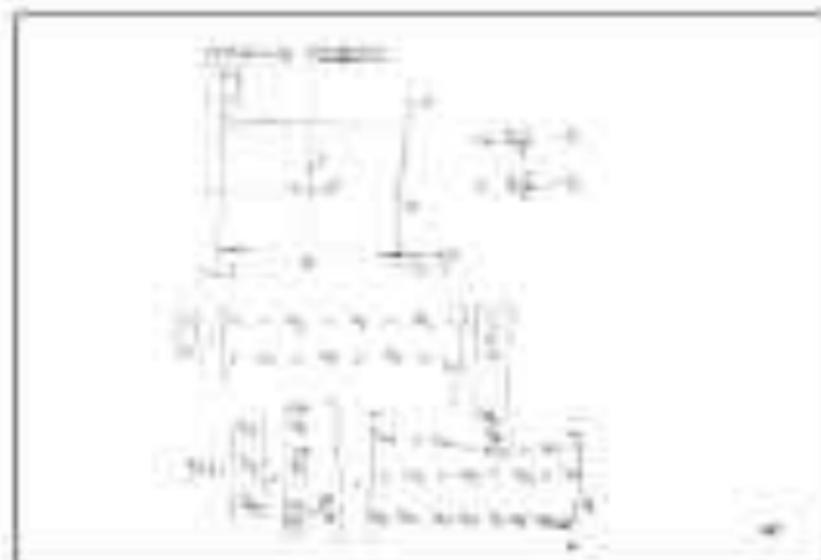
$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$

$$\text{Total strain energy} = \frac{1}{2} P \Delta = 1000000$$



$$\begin{aligned}
 u_1 &= \frac{F_1 L}{2E} + \frac{M_1 L^2}{2EI} + \frac{F_2 L}{2E} \\
 v_1 &= \frac{F_1 L^2}{2EI} + \frac{M_1 L}{EI} \\
 \theta_1 &= -\frac{F_1 L}{EI} + \frac{M_1}{EI} \\
 u_2 &= \frac{F_2 L}{2E} + \frac{M_2 L^2}{2EI} + \frac{F_1 L}{2E} \\
 v_2 &= \frac{F_2 L^2}{2EI} + \frac{M_2 L}{EI} \\
 \theta_2 &= \frac{F_2 L}{EI} + \frac{M_2}{EI}
 \end{aligned}$$

$$\begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 12 & -6L \\ 6L & 4L^2 & -6L & 2L^2 \\ 12 & -6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2
 \end{bmatrix}$$

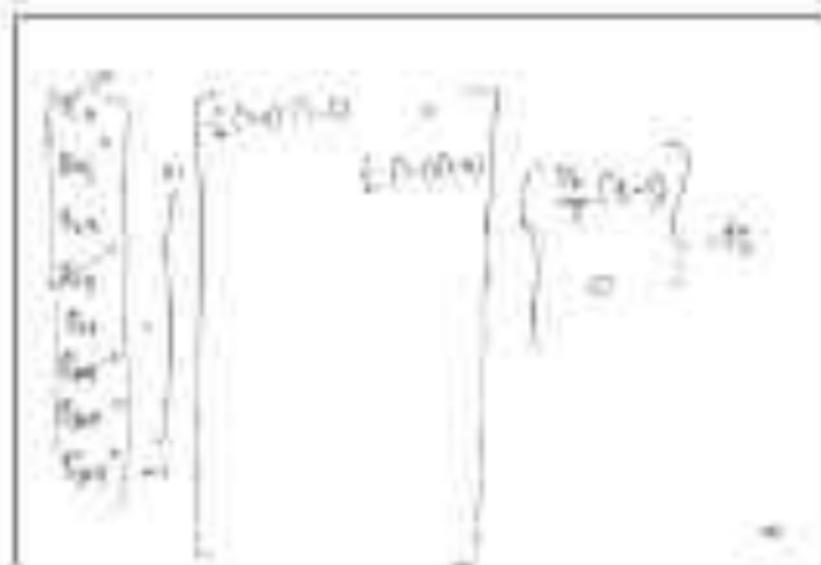
$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2
 \end{bmatrix}$$

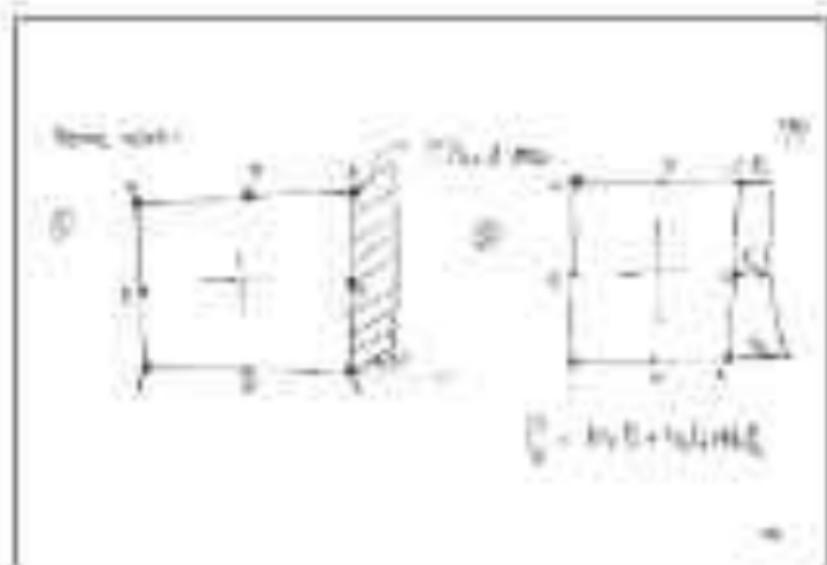
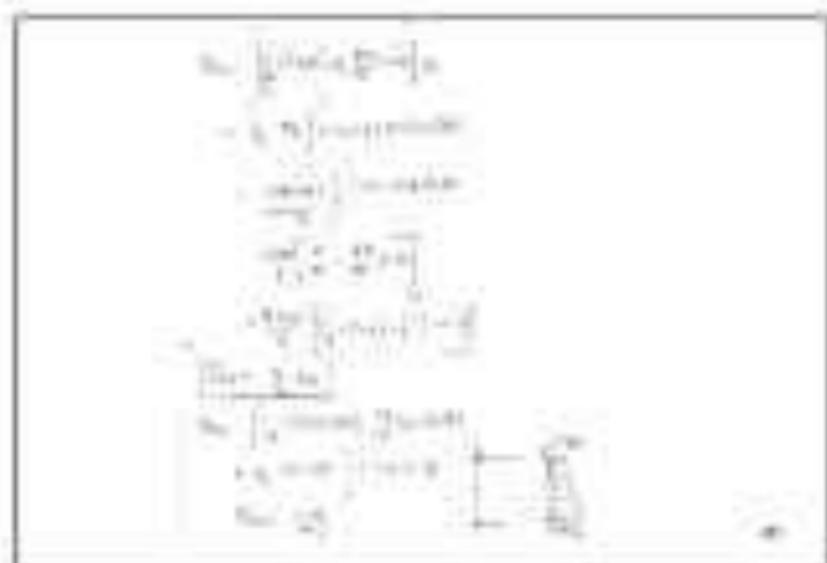
$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2
 \end{bmatrix}$$

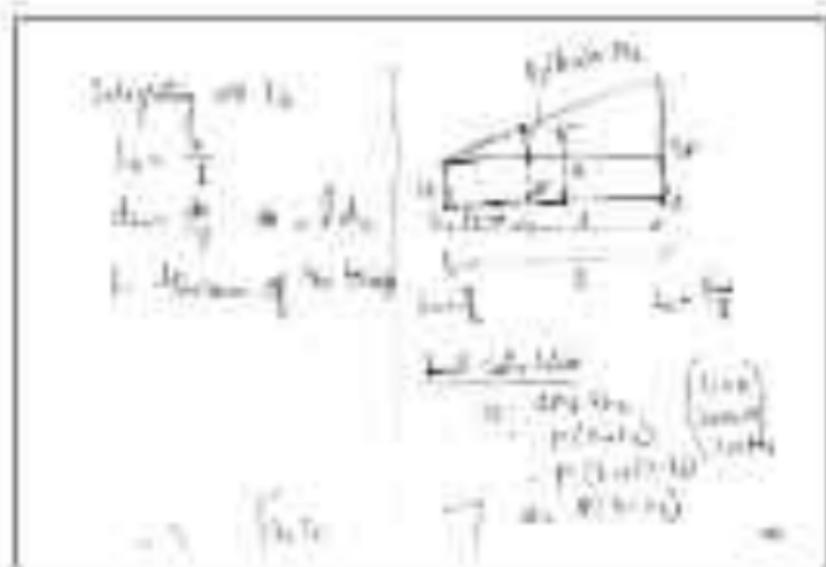
$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2
 \end{bmatrix}$$

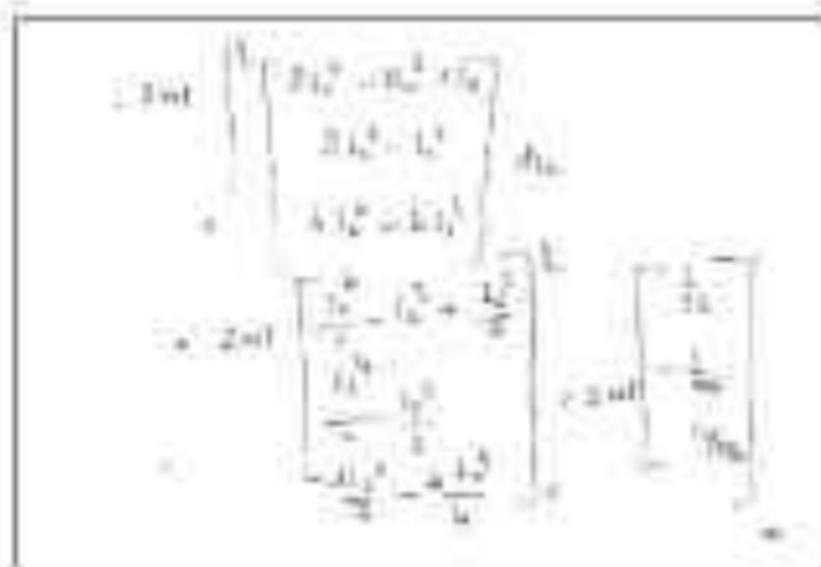
$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2
 \end{bmatrix}$$

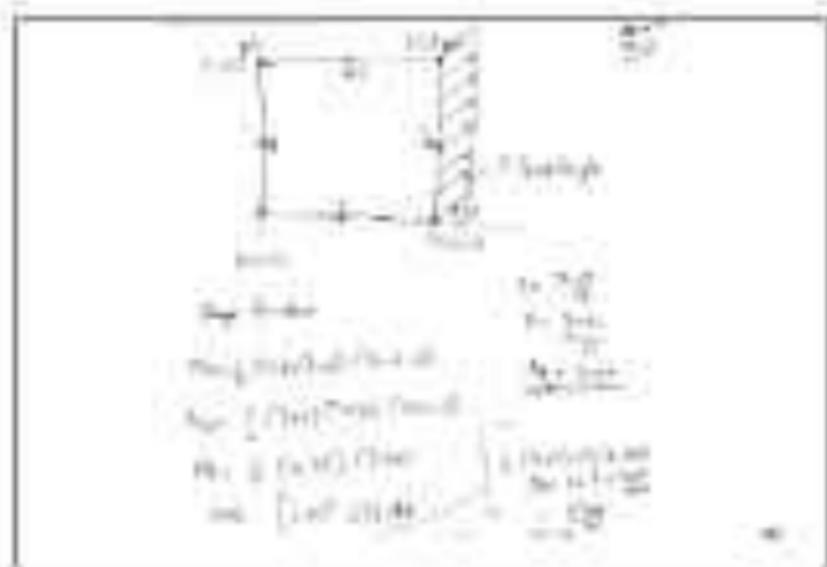
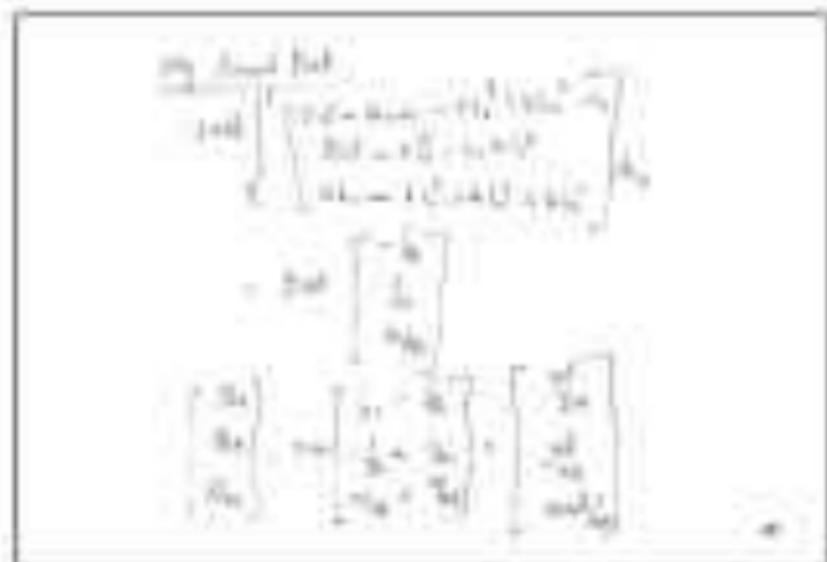
$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{bmatrix}
 u_1 \\
 v_1 \\
 \theta_1 \\
 u_2
 \end{bmatrix}
 =
 \begin{bmatrix}
 F_1 \\
 F_2 \\
 M_1 \\
 F_2
 \end{bmatrix}$$





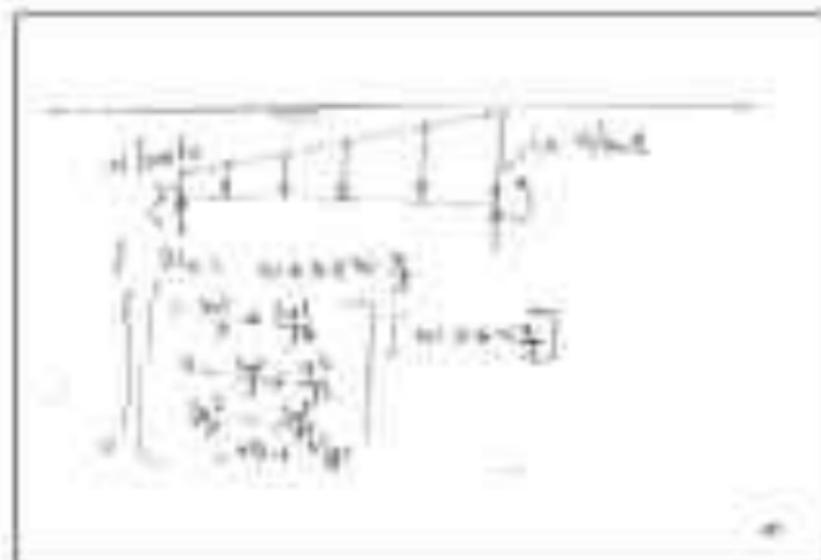






$$\begin{aligned}
 \text{Eqn 1} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix} \\
 \text{Eqn 2} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ P \end{bmatrix} \\
 \text{Eqn 3} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P \\ P \end{bmatrix} \\
 \text{Eqn 4} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P \\ -P \end{bmatrix} \\
 \text{Eqn 5} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -P \end{bmatrix} \\
 \text{Eqn 6} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -P \\ 0 \end{bmatrix} \\
 \text{Eqn 7} & \rightarrow \begin{bmatrix} \frac{EA}{L} & 0 \\ 0 & \frac{EA}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -P \\ -P \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{Eqn 1} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_2 - u_1) \\
 \text{Eqn 2} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_1 - u_2) \\
 \text{Eqn 3} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_2 - u_1) \\
 \text{Eqn 4} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_1 - u_2) \\
 \text{Eqn 5} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_2 - u_1) \\
 \text{Eqn 6} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_1 - u_2) \\
 \text{Eqn 7} & \rightarrow \text{Axial force} \\
 & = \frac{EA}{L} (u_2 - u_1)
 \end{aligned}$$



Isoparametric Element Formulation

Isoparametric Elements and Solution

- Digital breakthrough in the implementation of the finite element method is the development of an isoparametric element with capabilities to model structure problems geometries of any shape and size.
- The whole idea works in mapping:
 - The element in the real structure is mapped to an imaginary element in an ideal coordinate system.
 - The solution to the stress analysis problem is easy and known for the imaginary element.
 - These solutions are mapped back to the element in the real structure.
 - All the loads and boundary conditions are also mapped from the real to the imaginary element in the approach.

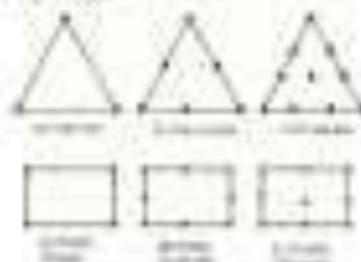
47

Isoparametric Formulation

Same technique that is used to define the element geometry is used to define the displacements within the element.

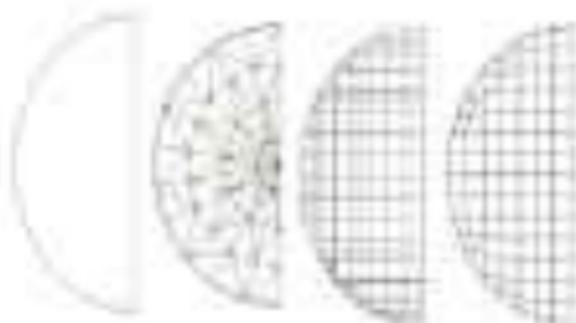
Necessity of Isoparametric Formulation

The use of 2-D and 3-D elements demands the use of irregular geometries i.e. triangles and rectangles instead of having straight edges.



48

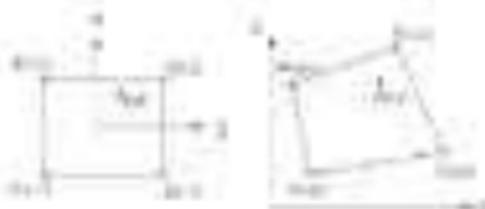
Analysis of any regular geometry is difficult to use with element shapes.



No problems in practice

40

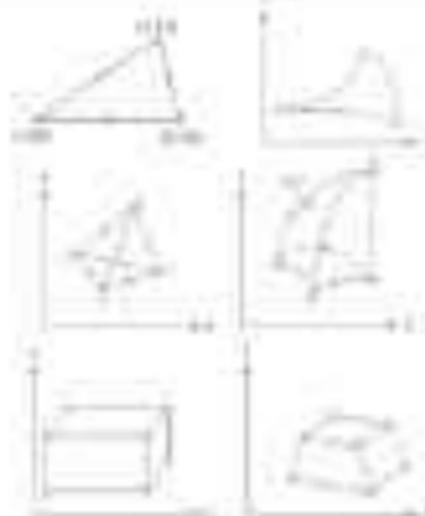
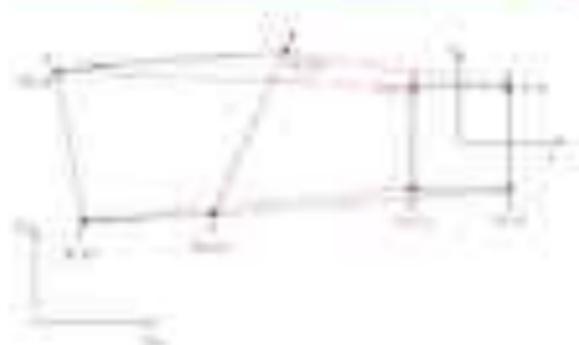
The finite element method works but we believe that using the elements along the contour of modeling.

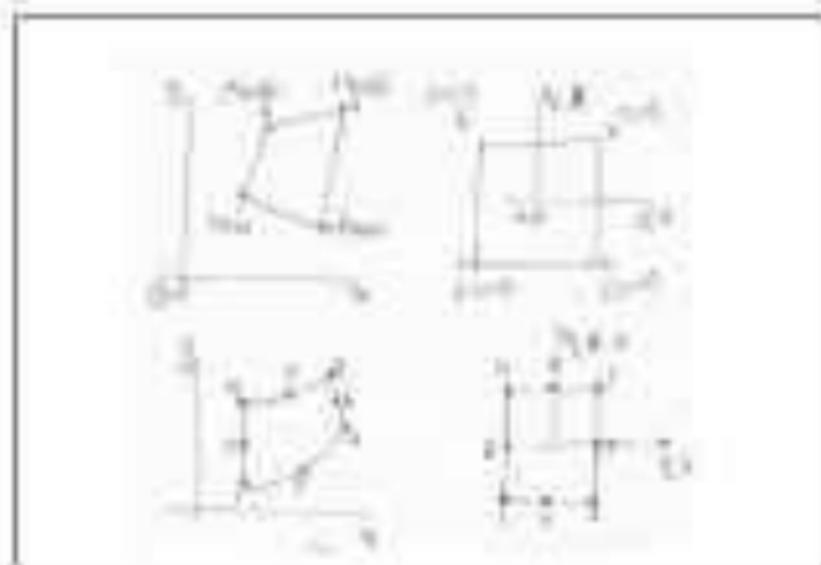
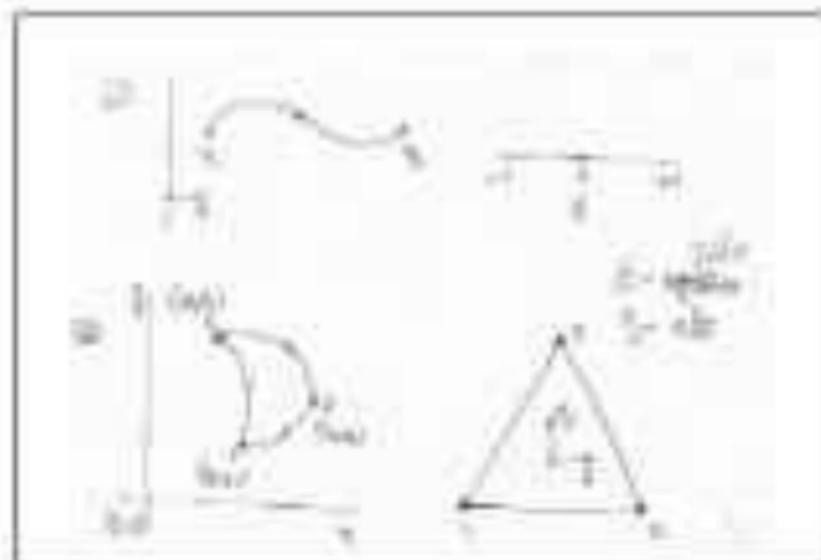


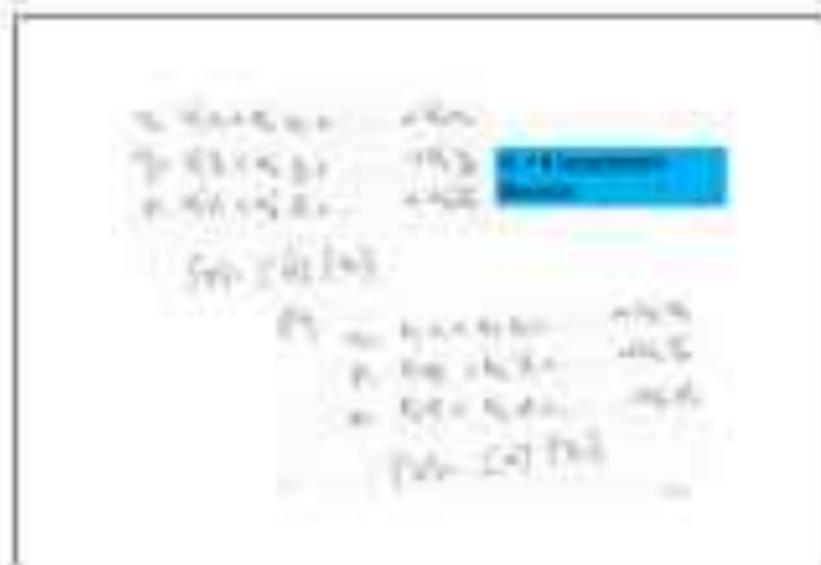
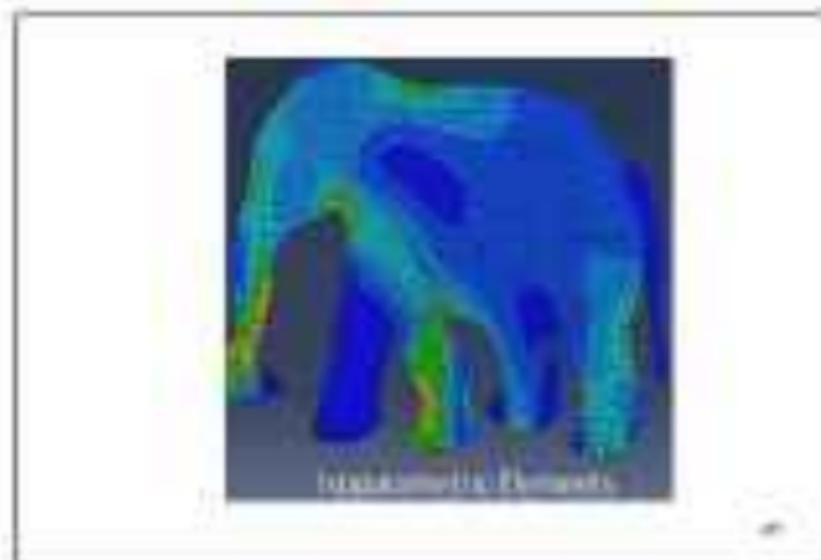
Using the concept of mapping regular triangles, rectangles or other elements in natural coordinate system (based on parent element) can be transformed into global Cartesian coordinate system having arbitrary shapes with curved edges or surfaces.

41

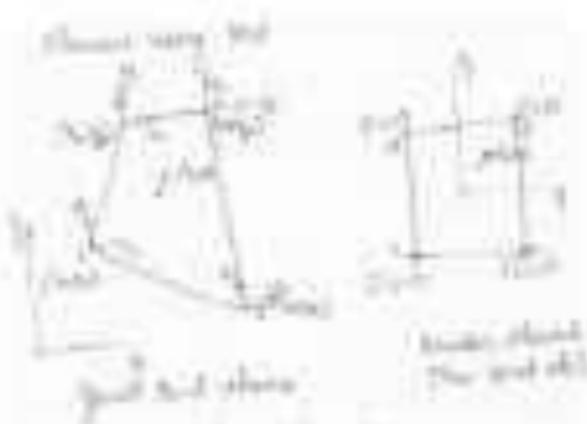
Isoparametric Element







Identification of the MDOFs of a frame structure
 around the y-axis



$\begin{matrix} \text{Node 1} \\ \text{Node 2} \\ \text{Node 3} \\ \text{Node 4} \end{matrix}$

The displacement of the structure is

$\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$

and the displacement of the structure is

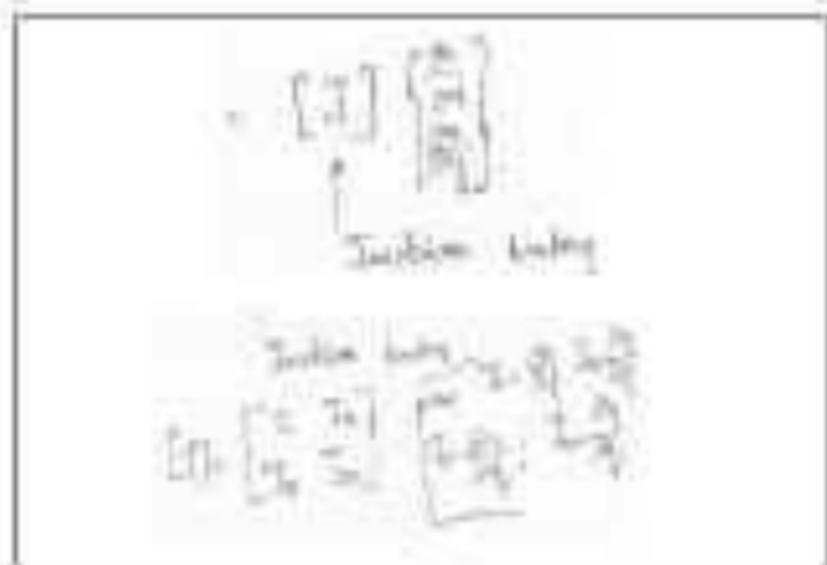
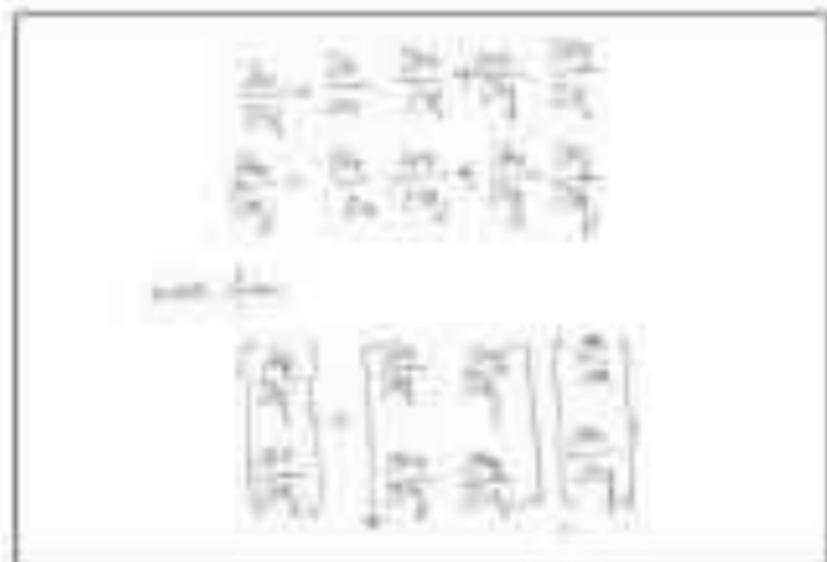
$\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}$

The rate of change of the quantity Q with respect to time t is denoted by \dot{Q} .

$$\dot{Q} = \frac{dQ}{dt}$$

$$\dot{Q} = \frac{dQ}{dt}$$

The rate of change of the quantity Q with respect to time t is denoted by \dot{Q} . The rate of change of the quantity Q with respect to time t is denoted by \dot{Q} . The rate of change of the quantity Q with respect to time t is denoted by \dot{Q} . The rate of change of the quantity Q with respect to time t is denoted by \dot{Q} .



$$\begin{aligned}
 \text{Node 1} & \rightarrow \text{Left} & \text{Node 2} & \rightarrow \text{Right} & \text{Node 3} & \rightarrow \text{Right} \\
 \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} \\
 \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} \\
 \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} \\
 \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2} & \frac{d^2 u}{dx^2} &= \frac{d^2 u}{dx^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Node 1} & \rightarrow \text{Left} \\
 \text{Node 2} & \rightarrow \text{Left} \\
 \text{Node 3} & \rightarrow \text{Left} \\
 \text{Node 4} & \rightarrow \text{Left} \\
 \text{Node 5} & \rightarrow \text{Left}
 \end{aligned}$$

Differential Eq. not given.

$$\begin{aligned}
 & \left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{array} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\
 & \Rightarrow \frac{1}{2} \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ 2u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\
 & \Rightarrow \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ 2u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 2 \end{Bmatrix} \\
 & \Rightarrow \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\
 & \Rightarrow \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Let } \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\
 & \Rightarrow \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\
 & \Rightarrow \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\
 & \Rightarrow \begin{Bmatrix} u_1 + u_2 \\ u_1 - u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}
 \end{aligned}$$

$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 \\
 0 & 0 & 0 & \frac{EA}{L}
 \end{bmatrix}
 \begin{Bmatrix}
 u \\
 v \\
 \theta \\
 w
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 P \\
 Q \\
 M \\
 R
 \end{Bmatrix}$$

(1)

-- [K] = [K] [B]
 -- [K] [B]
 -- [K] [B] [B]
 -- [K] [B] [B]
 -- [K] [B] [B]
 -- [K] [B] [B]

for Plane Stress Condition

$$E_{eff} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

for Plane Strain Condition

$$E_{eff} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\sigma_x, \sigma_y, \tau_{xy} \quad \epsilon_x = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$d_{11} = \int_{-a}^a \int_{-b}^b [x^2] \rho(x, y) dx dy$$

$$= \int_{-a}^a \int_{-b}^b [x^2] \rho_0 dx dy$$

$$= \rho_0 \int_{-a}^a \int_{-b}^b x^2 dx dy$$

$$= \rho_0 \int_{-a}^a \left[\frac{x^3}{3} \right]_{-b}^b dy$$

$$= \rho_0 \int_{-a}^a \left[\frac{b^3}{3} - \frac{(-b)^3}{3} \right] dy$$

$$= \rho_0 \int_{-a}^a \left[\frac{b^3}{3} + \frac{b^3}{3} \right] dy$$

$$= \rho_0 \int_{-a}^a \left[\frac{2b^3}{3} \right] dy$$

$$= \rho_0 \left[\frac{2b^3}{3} y \right]_{-a}^a$$

$$= \rho_0 \left[\frac{2b^3}{3} (a - (-a)) \right]$$

$$= \rho_0 \left[\frac{2b^3}{3} (2a) \right]$$

$$= \rho_0 \left[\frac{4ab^3}{3} \right]$$

Determine the centroidal coordinate of part of area of \square with base 2 , height 1 and $\rho = 1$ in the figure.



$$\begin{aligned}
 \text{So, } \frac{1}{2} (1+2) (1) &= \frac{1}{2} (1+2) (1-0) = 0.5 \\
 \text{So, } \frac{1}{2} (1+0) (1) &= \frac{1}{2} (1+0) (1-0) = 0.25 \\
 \text{So, } \frac{1}{2} (2+0) (1) &= \frac{1}{2} (2+0) (1-0) = 0.5 \\
 \text{So, } \frac{1}{2} (1+0) (1) &= \frac{1}{2} (1+0) (1-0) = 0.25
 \end{aligned}$$

particular value is

$$\begin{aligned}
 \text{So, } \frac{1}{2} (1+0) (1) &= \frac{1}{2} (1+0) (1-0) = 0.25 \\
 \text{So, } \frac{1}{2} (2+0) (1) &= \frac{1}{2} (2+0) (1-0) = 0.5 \\
 \text{So, } \frac{1}{2} (1+0) (1) &= \frac{1}{2} (1+0) (1-0) = 0.25
 \end{aligned}$$

So, $\frac{1}{2} (1+0) (1)$

$$\begin{aligned}
 &= \frac{1}{2} (1+0) (1-0) = 0.25 \\
 &= 0.25 \text{ unit}
 \end{aligned}$$

The constant value of part is $\frac{1}{2} (1+0) (1)$

Find cartesian coordinate of the point defined by
 (r, θ) as $(2.5, 0.5)$



The cartesian coordinate of point P
 $(2.42, 1.12)$

$$\begin{aligned}
 x &= r \cos(\theta) \\
 &= (2.5) \cos(0.5) \\
 &= 2.42 \\
 y &= r \sin(\theta) \\
 &= (2.5) \sin(0.5) \\
 &= 1.12
 \end{aligned}$$

For the transformation given below shown in the figure, find the local
 coordinate of point P where the cartesian coordinate is $(2, 4)$



$$2 \int_0^1 (1-x)^2 dx + \int_0^1 (1-x)^2 dx + \int_0^1 (1-x)^2 dx$$

$$= 3 \int_0^1 (1-x)^2 dx$$

$$= 3 \int_0^1 (1-x)^2 dx \quad \text{--- (1)}$$

$$= 3 \int_0^1 (1-x)^2 dx = 3 \quad \text{--- (2)}$$

$$\therefore \int_0^1 (1-x)^2 dx = 1$$

$$= \int_0^1 (1-x)^2 dx + \int_0^1 (1-x)^2 dx + \int_0^1 (1-x)^2 dx$$

$$= 3 \int_0^1 (1-x)^2 dx \quad \text{--- (3)}$$

$$2x + y = 0$$

$$2x - y = 0$$

$$4x = 0 \Rightarrow x = 0$$

$$2(0) + y = 0 \Rightarrow y = 0$$

$$\therefore \text{The solution is } (0, 0)$$

$$\text{Since the value of } x \text{ and } y \text{ is } 0$$

$$\therefore \text{The solution is } (0, 0)$$

Since the value of x and y ranges from -1 to 1 , the L.C.M. of points may be taken as $(-1, 1)$.

Find also the L, η values at the residual location (2, 1)

Repeat step ② and get value $\eta = 0$

$$z = z_0 + \eta \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \eta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 2 + \eta \\ 1 + \eta \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{--- ②}$$

and get $\eta = 0$

$$z = z_0 + \eta \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{--- ③}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{--- ④}$$

For the value of $\eta = 0$

the value of L is $L = 2$

and $\eta = 0$

the value of L is $L = 2$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \eta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the value of L is $L = 2$

and $\eta = 0$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Analyze the (a) and (b) global contributions for $e=1$ and the linear displacement vectors \mathbf{u}^e at joint conditions as shown in the figure.

(a)

$$[k] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

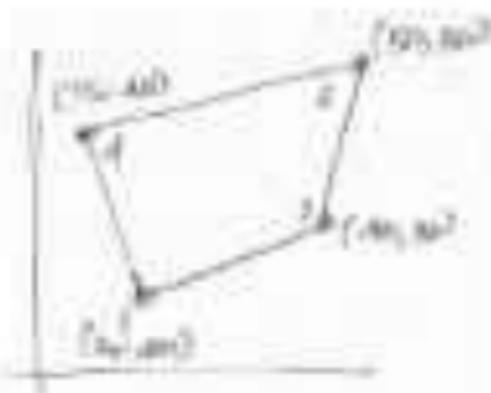
(b)

$$[k] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$[k] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$


Find the Jacobian at point (11, 12)



$$\begin{aligned}
 \mathbb{K} &= \frac{1}{2} \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (\mathbf{D} = \mathbf{D}^T) \\
 \mathbb{K} &= \frac{1}{2} \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (\mathbf{D} = \mathbf{D}^T) \\
 \mathbb{K} &= \frac{1}{2} \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (\mathbf{D} = \mathbf{D}^T) \\
 \mathbb{K} &= \frac{1}{2} \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (\mathbf{D} = \mathbf{D}^T) \\
 \mathbb{K} &= \frac{1}{2} \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \quad (\mathbf{D} = \mathbf{D}^T)
 \end{aligned}$$


The diagram shows a quadrilateral element with nodes labeled 1, 2, 3, and 4. A vertical line passes through node 1, extending from the top edge to the bottom edge. The element is shaded with a light gray background.

Teil 2

Teil 3

Teil 4

$$\mathbb{K} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ die Werte λ (Eigenwerte) sind $\lambda = \det(\mathbb{K} - \lambda \mathbb{I}) = 0$

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(1-\lambda) = 0$$

For a rectangular element shown in figure, determine the following at $t = 0.001175$.

- (a) Global Matrix
 (b) Global displacement matrix
 (c) Element Stress
 (d) Element Strain

Take $E = 200 \text{ GPa}$
 Assume Plane stress condition



Assume the nodal displacements $\{d\} = \{0, 0, 0.001, 0.001, 0.001, 0.001, 0, 0\}$

EFG: 4-nod. (Quadr)

$$\{d\} = \begin{bmatrix} 0 \\ 0 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0 \\ 0 \end{bmatrix}$$

$$\{d\} = \begin{bmatrix} 0 \\ 0 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0 \\ 0 \end{bmatrix}$$

$$\{d\} = \begin{bmatrix} 0 \\ 0 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0 \\ 0 \end{bmatrix}$$

$$\{d\} = \begin{bmatrix} 0 \\ 0 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0 \\ 0 \end{bmatrix}$$

$$\{d\} = \begin{bmatrix} 0 \\ 0 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0.001 \\ 0 \\ 0 \end{bmatrix}$$

Q. 10. Find element shape

$$\text{For } \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \text{I.E.C.E.C.} \quad \text{with displacement control}$$

$$\text{For } \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} \dots \\ \dots \end{Bmatrix}$$

$$\text{For } \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} \dots \\ \dots \\ \dots \end{Bmatrix} \text{ (matrix)}$$

Q. 11. Find element shape

$$\text{For } \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \text{I.E.C.E.C.}$$

For I.E.C. : $\int_{-1}^1 \int_{-1}^1 \dots \text{ (matrix) } \dots$
For plane stress condition

$$\begin{aligned}
 \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\cos^2 \theta} \end{bmatrix} \\
 = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\cos^2 \theta} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 [k] = [k]^{(1)} \\
 \{u\} = \{u\}^{(1)} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

Find the nodal displacement results for the frame structure shown in figure at the nodes given: u_1 (1.277 mm), u_2 (17.130 mm), u_3 (0 mm)



$$\text{Node } 1 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\text{Node } 2 \rightarrow \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$$

$$\text{Node } 3 \rightarrow \begin{bmatrix} 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \end{bmatrix}$$

$$\text{Node } 4 \rightarrow \begin{bmatrix} 25 & 26 & 27 & 28 & 29 \\ 30 & 31 & 32 & 33 & 34 \\ 35 & 36 & 37 & 38 & 39 \end{bmatrix}$$

$$\text{Node } 5 \rightarrow \begin{bmatrix} 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 & 51 & 52 & 53 & 54 & 55 & 56 & 57 & 58 & 59 & 60 & 61 & 62 & 63 & 64 & 65 & 66 & 67 & 68 & 69 & 70 & 71 & 72 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 & 81 & 82 & 83 & 84 & 85 & 86 & 87 & 88 & 89 & 90 & 91 & 92 & 93 & 94 & 95 & 96 & 97 & 98 & 99 & 100 \end{bmatrix}$$

$$\text{Node } 6 \rightarrow \begin{bmatrix} 101 & 102 & 103 & 104 & 105 & 106 & 107 & 108 & 109 & 110 & 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 & 120 & 121 & 122 & 123 & 124 & 125 & 126 & 127 & 128 & 129 & 130 & 131 & 132 & 133 & 134 & 135 & 136 & 137 & 138 & 139 & 140 & 141 & 142 & 143 & 144 & 145 & 146 & 147 & 148 & 149 & 150 & 151 & 152 & 153 & 154 & 155 & 156 & 157 & 158 & 159 & 160 & 161 & 162 & 163 & 164 & 165 & 166 & 167 & 168 & 169 & 170 & 171 & 172 & 173 & 174 & 175 & 176 & 177 & 178 & 179 & 180 & 181 & 182 & 183 & 184 & 185 & 186 & 187 & 188 & 189 & 190 & 191 & 192 & 193 & 194 & 195 & 196 & 197 & 198 & 199 & 200 \end{bmatrix}$$

Coordinate Transformation

The geometry of an element may be expressed in terms of the interpolation functions as follows:

$$\begin{aligned}
 x &= N_1 x_1 + N_2 x_2 + \dots + N_n x_n = \sum_{i=1}^n N_i x_i \\
 y &= N_1 y_1 + N_2 y_2 + \dots + N_n y_n = \sum_{i=1}^n N_i y_i \\
 z &= N_1 z_1 + N_2 z_2 + \dots + N_n z_n = \sum_{i=1}^n N_i z_i
 \end{aligned}$$

- where
- 1) x_i, y_i, z_i are the coordinates of the nodes
 - 2) N_i are the interpolation functions
 - 3) i, j, k are the indices of the nodes of the element

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One can also express the field variable variation in the element as

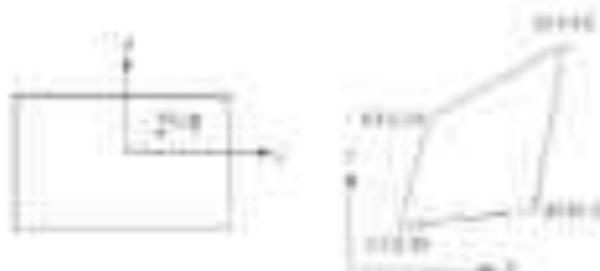
$$\phi(\bar{x}, \bar{y}, \bar{z}) = \sum_{i=1}^n N_i(\bar{x}, \bar{y}, \bar{z}) \phi_i$$

As the same shape functions are used for both the field variable and description of element geometry, the method is known as isoparametric mapping.

The element defined by such a method is known as an isoparametric element.

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Determine the Cartesian coordinates of the point P ($x = 0.8$, $y = 0.6$) as shown in Fig.



As mentioned above, the natural (local) and Cartesian coordinates can be expressed through their respective isoparametric functions. Therefore, the values of the isoparametric function at user P will be

$$\begin{aligned} \xi &= \frac{0(-1) + 0(1) + 0.6(1) + 0(-1)}{4} = 0.15 \\ \eta &= \frac{0(1) + 0(-1) + 0.6(1) + 0(-1)}{4} = 0.15 \\ \xi &= \frac{0(1) + 0(-1) + 0.6(1) + 0(-1)}{4} = 0.15 \\ \eta &= \frac{0(-1) + 0(1) + 0.6(1) + 0(-1)}{4} = 0.15 \end{aligned}$$

Thus the coordinates of user P in Cartesian coordinate system can be calculated as

$$\begin{aligned} x &= \sum_{i=1}^4 N_i x_i = 0.15(0) + 0.15(1) + 0.15(0.8) + 0.15(0) = 0.27 \\ y &= \sum_{i=1}^4 N_i y_i = 0.15(0) + 0.15(0) + 0.15(0.6) + 0.15(1) = 0.27 \end{aligned}$$

Thus the natural coordinates of P is (ξ, η) from Cartesian coordinate system are $(0.27, 0.27)$.

Concept of Jacobian Matrix

- A variety of derivatives of the interpolation functions with respect to the global coordinates are necessary to formulate the element stiffness matrix.
- As the both element geometry and variation of the shape functions are represented in terms of the natural coordinates of the parent element, some additional mathematical obstacle arises.
- For example, in case of evaluation of the strain vector, the spatial matrix is with respect to x and y , but the interpolation function is with ξ and η .

The connectivity between the coordinate systems has to be established by using the chain rule of partial differentiation as

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta}$$

The above equations can be expressed in matrix form as well,

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{pmatrix}$$

The matrix J is termed as Jacobian matrix, which is

$$J = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix}$$

$$\text{As we know, } v = \sum_{i=1}^n N_i x_i$$

where, n is the number of nodes in an element

$$x_i = \frac{\partial v}{\partial x} = \frac{\partial \sum_{j=1}^n N_j x_j}{\partial x} = \sum_{j=1}^n \frac{\partial N_j}{\partial x} x_j$$

Substituting in above for the term T_1 and T_2 for further work.

Then,

$$T_1 = \left[\frac{\partial N_1}{\partial x} - \frac{\partial N_2}{\partial x} \right] x_1 - \left[\frac{\partial N_1}{\partial x} - \frac{\partial N_2}{\partial x} \right] x_2$$

+

and similarly

$$\left[\frac{\partial N_3}{\partial x} - \frac{\partial N_4}{\partial x} \right] x_3 - \left[\frac{\partial N_3}{\partial x} - \frac{\partial N_4}{\partial x} \right] x_4$$

Considering $\left[\frac{\partial N_i}{\partial x} \right]$ as the matrix of nodal B-matrix.

we have,

$$\frac{\partial}{\partial x} = J_1 \frac{\partial}{\partial \xi} + J_2 \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} = J_3 \frac{\partial}{\partial \xi} + J_4 \frac{\partial}{\partial \eta}$$

+

Similarly, the first element can be

$$\begin{aligned}
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
 \end{aligned}$$

where

$$\begin{aligned}
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 & \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
 \end{aligned}$$

4

Evaluation of Stiffness Matrix of 2-D Isoparametric Elements

For two dimensional plane stress/strain formulation, the strain vector can be represented as

$$\begin{aligned}
 \{\epsilon\} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} &= \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n \frac{\partial N_i}{\partial x} u_i \\ \sum_{i=1}^n \frac{\partial N_i}{\partial y} v_i \\ \sum_{i=1}^n \left(\frac{\partial N_i}{\partial y} u_i + \frac{\partial N_i}{\partial x} v_i \right) \end{bmatrix}
 \end{aligned}$$

5

The finite approximation to the problem is written as

$$\mathbf{K} \mathbf{u} = \mathbf{F}$$

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

For a 1D problem the displacement can be approximated as

$$u = \sum_{i=1}^n N_i u_i \quad \text{substituted to (1)}$$

Then

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
 \frac{1}{E} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} &= \begin{bmatrix} \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1-\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1-\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1-\nu) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \\
 \sigma &= E \mathbf{B} \epsilon
 \end{aligned}$$

Or,

$$\{\sigma\} = \mathbf{B} \{\epsilon\}$$

Where $\{\epsilon\}$ is the nodal displacement vector

\mathbf{B} is known as strain-displacement relation matrix and called as strain matrix.

$$\mathbf{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & -\frac{\partial}{\partial z} \end{bmatrix}$$

It is necessary to be clear primary from Cartesian to the natural coordinate as well for calculation of the element stiffness and other quantities for isoparametric formulation.

The differential area increment can be established from arbitrary values along the coordinate axes in Cartesian coordinate can be represented in terms of area in value of coordinates as

$$dA = dx dy = \left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| d\xi d\eta$$

Here $\left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right|$ is the Jacobian of the mapping matrix.

The stiffness matrix for a two-dimensional element may be expressed as

$$K = \iint_{\Omega} B^T D B \, d\Omega = \iint_{\Omega} B^T D B \, dx dy$$

Ω is the two dimensional element area

Ω is the domain of the element

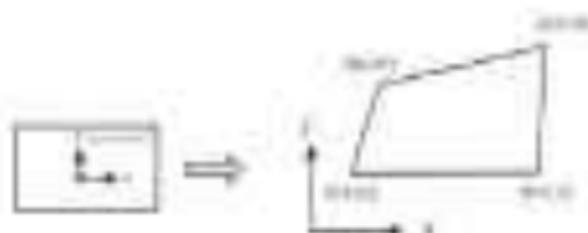
19

The above expression in Cartesian coordinate system can be changed to the natural coordinate system as follows by obtain the elemental stiffness matrix

$$K = \iint_{\Omega} B^T D B \, dx dy$$

20

Calculate the Jacobian matrix and the strain displacement matrix for four nodal rectangular elements, corresponding to the given point $(0.57735, 0.57735)$ as shown in Fig.



Two dimensional quadrilateral element

Solution:

The Jacobian matrix for a four node element is given by,

$$J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

Therefore the values are as follows for the given element

$$x_1 = 0, y_1 = 0, x_2 = 1, y_2 = 0$$

$$x_3 = 1, y_3 = 1, x_4 = 0, y_4 = 1$$

$$x_5 = 0, y_5 = 0, x_6 = 1, y_6 = 0$$

$$x_7 = 0, y_7 = 1, x_8 = 1, y_8 = 1$$

Then,

$$J_{11} = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial x_i}{\partial \xi} \frac{\partial x_j}{\partial \eta} = 0.1000 \cdot 2 + 0.4000 \cdot 1.5 = 0.2000 \text{ (1.7.1.1)}$$

Similarly, $J_{12} = 0.0000$, $J_{21} = 0.2500$ and $J_{22} = 1.0000$.

Thus,

$$J = \begin{bmatrix} 0.2000 & 0.0000 \\ 0.2500 & 1.0000 \end{bmatrix}$$

Thus, the areas of the surfaces will become

$$J_1 = \begin{vmatrix} J_1 & J_2 \\ J_3 & J_4 \end{vmatrix} = \begin{vmatrix} 1.000 & -0.000 \\ -0.000 & 1.000 \end{vmatrix}$$

□

Three-noded displacement with a gravity

$$K = \begin{bmatrix} C & C & 0 & 0 \\ 0 & 0 & C & C \\ C & C & C & C \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ C \\ C \end{vmatrix}$$

$$\begin{bmatrix} 1000 & 1000 & 0 & 0 \\ 0 & 0 & 1000 & 1000 \\ 1000 & 1000 & 1000 & 1000 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1000 & 1000 & 0 & 0 \\ 0 & 0 & 1000 & 1000 \\ 1000 & 1000 & 1000 & 1000 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1000 \\ 1000 \end{bmatrix}$$

□

Then the displacement can be given by,

$$U = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

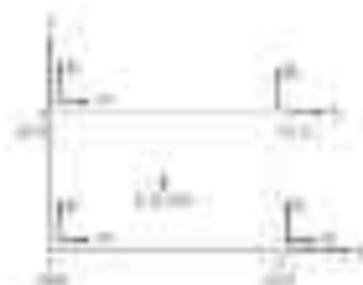
The stiffness matrix can be found by using the following expression in natural coordinate system,

$$K = \iint_{\Omega} B^T D B \, d\Omega = \iint_{\Omega} B^T D B \, dx \, dy$$

$$[\text{N} \times \text{N} \times \text{N} \times \text{N}]$$

$$K = \iint_{\Omega} B^T C B \, dx \, dy$$

Then, the element stiffness matrix is given by, $K^e = \int_{\Omega^e} B^T D B \, d\Omega$
 For the element e the displacement is given by, $U^e = [u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6 \ u_7 \ u_8]^T$



$$\begin{aligned}
 \Rightarrow & \begin{bmatrix} \frac{1+\nu}{4} & \frac{1-\nu}{4} & \frac{1+\nu}{4} & \frac{1-\nu}{4} \\ \frac{1-\nu}{4} & \frac{1+\nu}{4} & \frac{1-\nu}{4} & \frac{1+\nu}{4} \\ \frac{1+\nu}{4} & \frac{1-\nu}{4} & \frac{1+\nu}{4} & \frac{1-\nu}{4} \\ \frac{1-\nu}{4} & \frac{1+\nu}{4} & \frac{1-\nu}{4} & \frac{1+\nu}{4} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \\
 & = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 \mathbf{K} & = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & 0 & 0 \\ 0 & 0 & \mathcal{K}_{21} & \mathcal{K}_{22} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{31} & \mathcal{K}_{32} \end{bmatrix}
 \end{aligned}$$

$$\mathbf{K} = \frac{1}{\Delta L} \begin{bmatrix} \mathcal{K}_{11} & -\mathcal{K}_{12} & 0 & 0 \\ 0 & 0 & -\mathcal{K}_{21} & \mathcal{K}_{22} \\ -\mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{31} & -\mathcal{K}_{32} \end{bmatrix}$$

$$\mathbf{K} = \frac{1}{(L/2)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \Delta \mathbf{G}$$

$$\epsilon = \mathbf{B} \mathbf{u}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $\mathbf{u}^T \mathbf{u} = 0$

$$\mathbf{W} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

 The condition $\mathbf{u}^T \mathbf{u} = 0$ can be given by the product

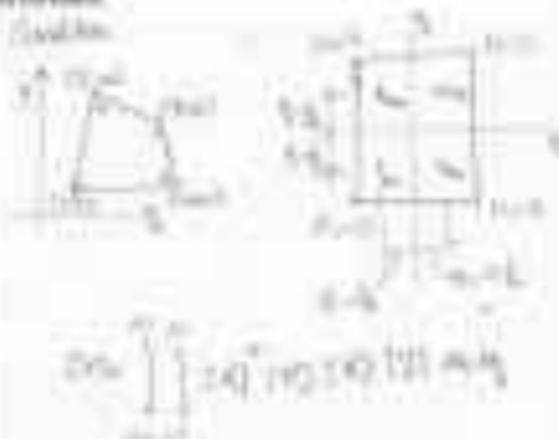
$$\mathbf{u}^T = \mathbf{W}^T \mathbf{q}$$

$$\begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \mathbf{q}$$

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} -q_1 & q_2 & q_3 \\ q_4 & -q_5 & q_6 \\ q_7 & q_8 & q_9 \end{bmatrix}$$

 Then $\mathbf{u}^T = \frac{1}{2} (q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6 \quad q_7 \quad q_8 \quad q_9)$

For the quadrilateral element shown in the figure, calculate the stiffness matrix assuming $E = 30000 \text{ N/mm}^2$ and Poisson's ratio = 0.25, a unit thickness and a plane stress condition.



Stiffness matrix

Node	1	2
1	$\frac{EA}{L}$	$-\frac{EA}{L}$
2	$-\frac{EA}{L}$	$\frac{EA}{L}$

(Equation 1)

$$\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$EA \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} FL \\ F_2 \end{Bmatrix}$$

$$EA \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} FL \\ F_2 \end{Bmatrix}$$

$$EA \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} FL \\ F_2 \end{Bmatrix}$$

Q4) Q4Q4

$$Q4 = \frac{1}{16} \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix}$$

$$Q4 = \frac{1}{16} \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

So the answer is 4-1000 and
 the rest of the steps

$$Q4 = \frac{1}{16} \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{16} \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{16} \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

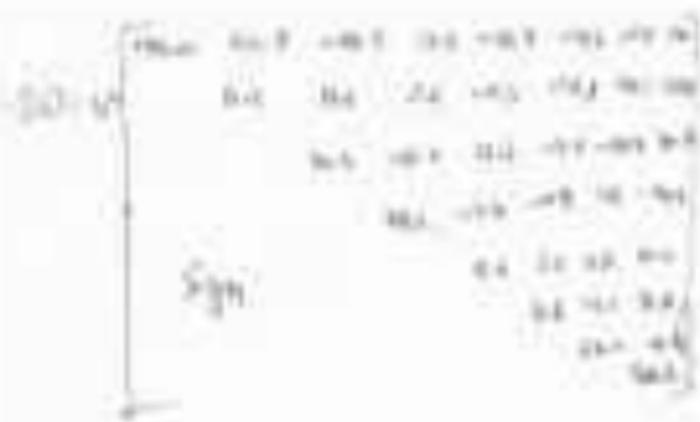
$$= \frac{1}{16} \begin{bmatrix} 25 & 15 & 5 & 1 \\ 15 & 25 & 1 & 5 \\ 5 & 1 & 25 & 15 \\ 1 & 5 & 15 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

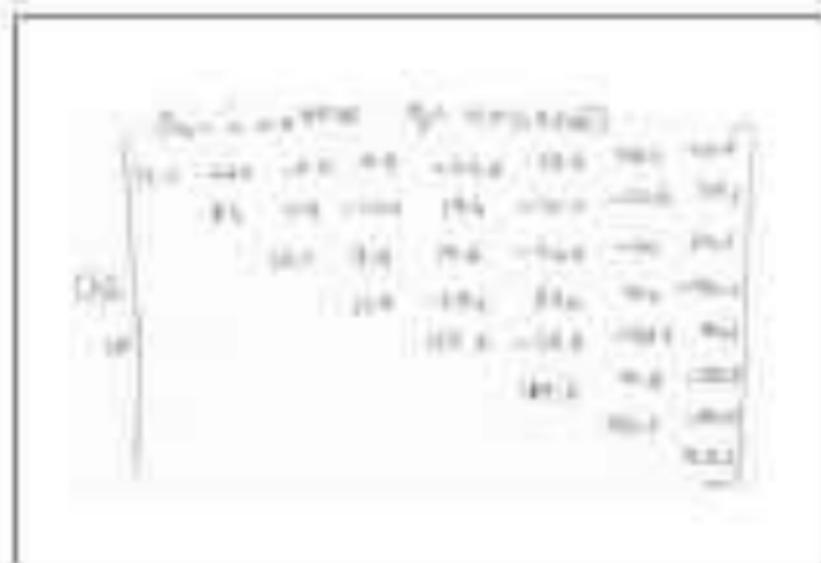
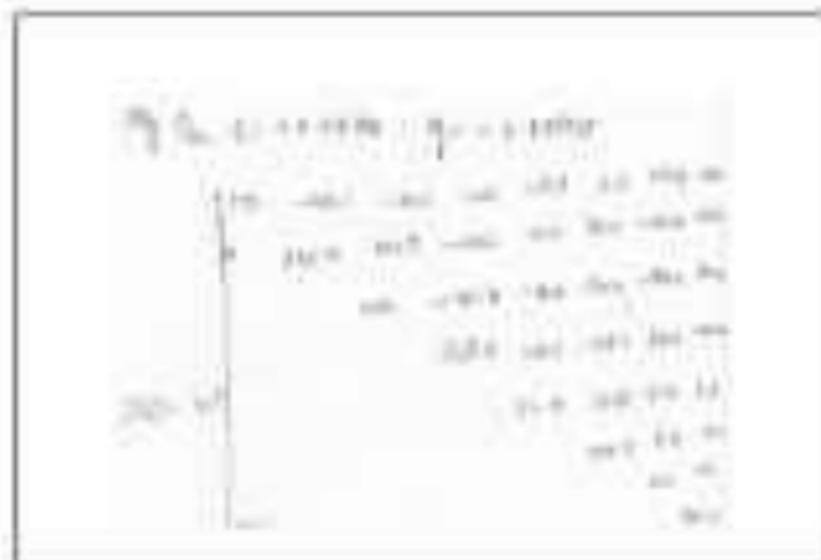
$\therefore \int_{\Omega} \nabla \cdot \mathbf{q} \, d\Omega = \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} \, d\Gamma$

$\therefore \int_{\Omega} \nabla \cdot \mathbf{q} \, d\Omega = \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} \, d\Gamma$

$\therefore \int_{\Omega} \nabla \cdot \mathbf{q} \, d\Omega = \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} \, d\Gamma$

$\therefore \int_{\Omega} \nabla \cdot \mathbf{q} \, d\Omega = \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} \, d\Gamma$





At the top of the column, the matrix

$$\begin{array}{c}
 \text{K}_{11} \\
 \left[\begin{array}{ccccccc}
 12EI & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 12EI & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 6EI & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 6EI & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 4EI & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 4EI & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 4EI
 \end{array} \right]
 \end{array}$$

The stiffness matrix for the column is

$$\begin{array}{c}
 \text{K}_{11} \\
 \left[\begin{array}{ccccccc}
 12EI & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 12EI & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 6EI & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 6EI & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 4EI & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 4EI & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 4EI
 \end{array} \right]
 \end{array}$$

Numerical Integration



Numerical Integration

In the finite element analysis, we face the problem of evaluating the following type of integrations in one, two, and three dimensional cases respectively.

These are necessary to compute element stiffness and element load vector

$$\int_{\Omega} f(x) dx, \int_{\Omega} f(x) y dx, \int_{\Omega} f(x) y z dx$$

Appropriate solutions to such problems are possible using certain numerical techniques.



Several numerical techniques are available in mathematics for solving definite integrative analysis, viz., Gauss, midpoint rule, trapezoidal rule, Simpson's 1/3rd rule, Simpson's 3/8th rule and Gauss Quadrature formula.

Among these, Gauss Quadrature formula is most useful for solving problems in finite element method.

GAUSS QUADRATURE FOR ONE-DIMENSIONAL ELEMENTS

The concept of Gauss Quadrature in the context of one dimension is the concept of an integral in the form of

$$\int_{-1}^{+1} f(\xi) d\xi \quad \text{Here } \int_{x_1}^{x_2} f(x) dx;$$

→

To evaluate the one dimensional integral of $x_1 \leq x \leq x_2$ in an interval of $-1 \leq \xi \leq 1$, we need to change the original domain from $f(x) = f(\xi)$ as follows.

$$x = \frac{1+\xi}{2} x_1 + \frac{1-\xi}{2} x_2 = \xi_1 x_1 + \xi_2 x_2$$

$$\text{where } \xi_1 = \frac{1-\xi}{2}, \xi_2 = \frac{1+\xi}{2}$$

$$\xi_1 + \xi_2 = 1, \quad \xi_2 - \xi_1 = \xi$$

$$\therefore \int_{x_1}^{x_2} f(x) dx = \int_{-1}^{+1} f(\xi) d\xi$$

→

Special case: find a function u over the whole Ω of interest
 on a grid $\{x_i\}$. Find a polynomial u_p that satisfies given
 boundary conditions

$$1 \rightarrow \int_{\Omega} u(x) dx = \sum_{i=1}^n w_i (u(x_i) + u(x_{i+1}) + \dots + u(x_{i+1}))$$

One-Point Formula

Consider $n = 1$.

$$\int_{\Omega} u(x) dx = w_1 u(x_1)$$



Two-Point Formula

If we consider $n = 2$,

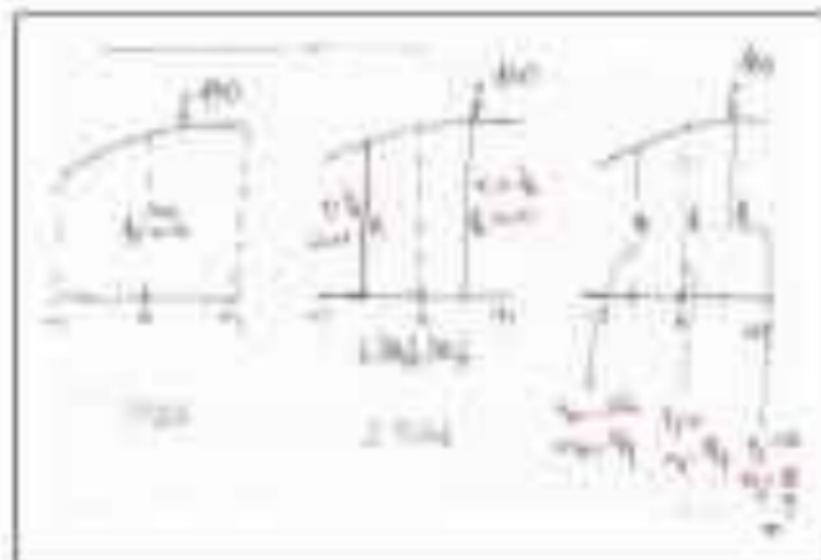
$$\int_{\Omega} u(x) dx = w_1 u(x_1) + w_2 u(x_2)$$

Case Quadrature for Two-Dimensional Integrals

$$1 \rightarrow \int_{\Omega} \int_{\Omega} u(x, y) dx dy =$$

$$1 \rightarrow \sum_{i=1}^n \sum_{j=1}^m w_i w_j u(x_i, y_j)$$





$$\int_{\Omega} \sigma_{ij} \epsilon_{ij} \, d\Omega$$
 is the energy of the system.

The total energy of the system is given by

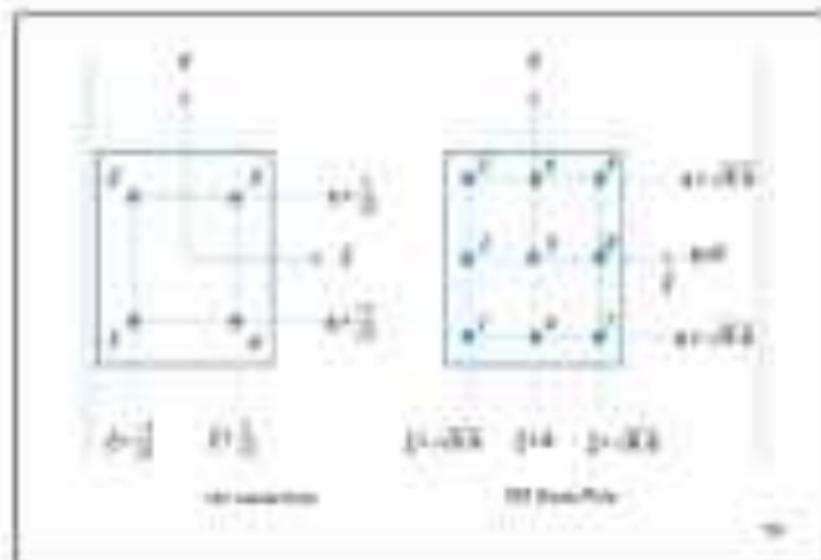
$$U = \int_{\Omega} \sigma_{ij} \epsilon_{ij} \, d\Omega - \int_{\Omega} b_i u_i \, d\Omega - \int_{\Gamma} t_i u_i \, d\Gamma$$

where b_i is the body force, t_i is the surface force, and u_i is the displacement.

The total energy of the system is a function of the nodal displacements u_n .

Node	Type	x	y	Shape Functions			Type
				N_1	N_2	N_3	
1	Node	0	0	1	0	0	Type Node Element
2	Node	1	0	0	1	0	
3	Node	0	1	0	0	1	
4	Node	1	1	0	0	0	

Figure 1: Shape Functions for a Triangle



If both sides of the differential equation are
 known quantities, then only a problem
 (boundary value problem) needs to be solved.
 The solution is given by $f(x) = \frac{1}{L} \int_0^L v(x) dx$
 done numerically. $\frac{1}{L} \int_0^L v(x) dx$ is the
 average value of $v(x)$ over the
 domain. $\frac{1}{L} \int_0^L v(x) dx$ is the
 average value of $v(x)$ over the
 domain. $\frac{1}{L} \int_0^L v(x) dx$ is the
 average value of $v(x)$ over the
 domain.

\Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 The area of the triangle is 50
 The area of the triangle is 50
 The area of the triangle is 50
 \Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 \Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 The area of the triangle is 50
 The area of the triangle is 50

\Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
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 \Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 \Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 The area of the triangle is 50
 The area of the triangle is 50
 \Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 \Rightarrow $\frac{1}{2} \times 10 \times 10 = 50$
 The area of the triangle is 50
 The area of the triangle is 50

We have $\int_{-1}^1 [x^2(x^2-1)] dx$ is zero as per
 the above theorem. Hence, the integral is zero.
 The integral is zero.
 $\int_{-1}^1 [x^2(x^2-1)] dx = 0$
 $\int_{-1}^1 [x^2(x^2-1)] dx = 0$
 The integral is zero.
 The integral is zero.
 The integral is zero.

Check - 1

$$\int_{-1}^1 \left(x^2 + \frac{1}{2} \right) dx = \left[\frac{x^3}{3} + \frac{x}{2} \right]_{-1}^1$$

$$= \left[\frac{1}{3} + \frac{1}{2} \right] - \left[-\frac{1}{3} - \frac{1}{2} \right]$$

$$= \frac{4}{6} + \frac{4}{6}$$

The value is $\frac{4}{3}$ or 1.3333

$$\text{Total } \int_{-1}^1 (5x^2 + 4x) dx$$

Area of 2 gauss points

$$2 \times 1 = 2$$

$$= 11 \text{ gauss}$$

Boundary of element P_1, P_2, P_3
 $= (1, 0, 0)$

2nd element

$$u_1 = 0, u_2 = 1$$

$$P_1 = (0, 1, 0), P_2 = (1, 1, 0)$$

$$P_3 = (0, 0, 1), P_4 = (1, 0, 1)$$

$$u = \int_{-1}^1 (5x^2 + 4x) dx = \left[\frac{5}{3}x^3 + 2x^2 \right]_{-1}^1 = \frac{5}{3} + 2 - \left(-\frac{5}{3} + 2 \right) = \frac{10}{3}$$

$$\text{Area of element } = \int_{-1}^1 (5x^2 + 4x) dx = \frac{10}{3}$$

$$= 10$$

$$\text{Max. of } \sigma_{xx} = \frac{w_0}{4} \left[\frac{1}{2} (1 - \cos 2\theta) - \frac{1}{2} (1 + \cos 2\theta) \right] \\ = -\frac{w_0}{4} (1 + \cos 2\theta)$$

$$\text{At } \theta = 0, \sigma_{xx} = -\frac{w_0}{4} [2 - 1 - 1] = 0$$

$$\text{Max. } \int_0^{\pi/2} \int_0^{\pi/2} \sigma_{xy} = 0 \text{ at } \theta = \pi/4$$

$$\text{Ex. } \int_0^{\pi/2} \int_0^{\pi/2} (x+y) dx dy$$

$$= \int_0^{\pi/2} \left[\frac{x^2}{2} + xy \right]_0^{\pi/2} dy$$

$$= \int_0^{\pi/2} \left[\frac{(\pi/2)^2}{2} + (\pi/2)y \right] dy$$

$$= \int_0^{\pi/2} \left(\frac{\pi^2}{8} + \frac{\pi y}{2} \right) dy$$

$$\left[\frac{\pi^2 y}{8} + \frac{\pi y^2}{4} \right]_0^{\pi/2} = \frac{\pi^3}{16} + \frac{\pi^3}{16} = \frac{\pi^3}{8}$$

Ans. $\frac{\pi^3}{8}$

Axis-symmetric Element Formulation

Axisymmetric Analysis



Three-dimensional analysis for 2 dimensional problem.

It is possible to solve the problem by two dimensional analysis using 2D elements with axisymmetric shell element. This can be analyzed using 2D elements.

If the same finite element concepts from an introductory course (in which of course, one thought of the three dimensions as completely fixed, they may be present using 2D elements due to their simplicity:

Examples

- Planar stress
- Cylinders loaded by uniform internal (or external) pressure
- Bubbles
- Turbine shrouts
- Circular cooling rings (in solid rocket case)

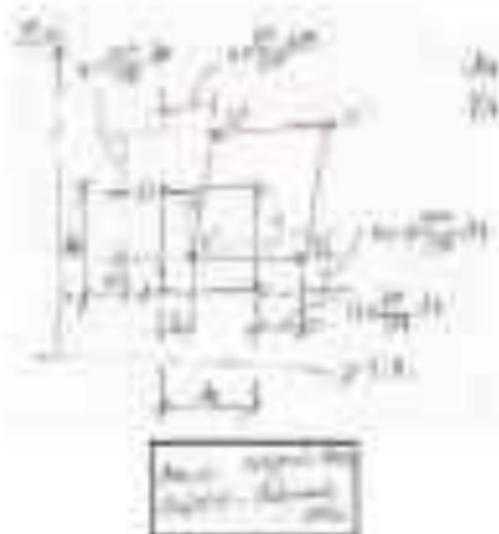
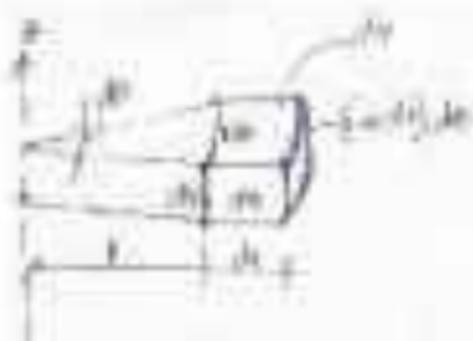


In the parametric analysis, the respective (in stress and deformation of the total analysis of the body of interest) path respect to symmetry are **difficult**.

Usually, structural conditions $\sigma = 0$ and $u = 0$ used in the analysis. Because of symmetry, all the nodal degrees and stresses are **independent** of symmetry Σ .

Thus the problem is reduced to Ω in the coordinate x and y .



Formulation


Q1) Find the value of $\frac{dy}{dx}$ at $x=1$ by

$$f_1 = \frac{(1+2x^2) - 1}{2} = \frac{2}{2} = 1 \quad \text{--- (1)}$$

Q2) Find the value of $\frac{dy}{dx}$ at $x=1$ by

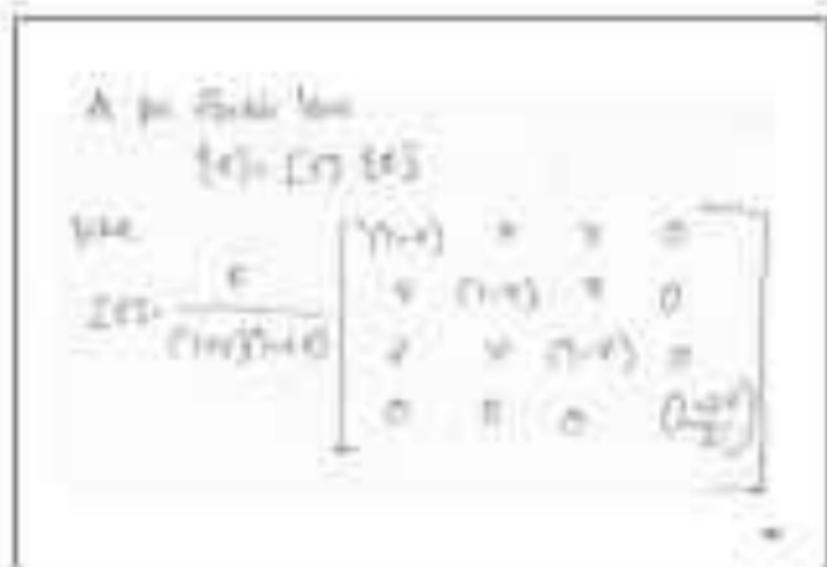
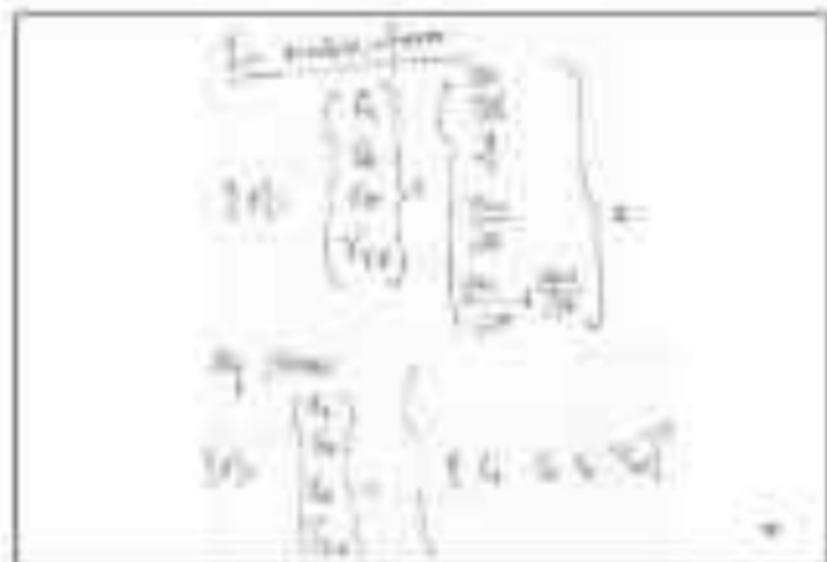
$$f_2 = \frac{(1+2x^2) - 1}{2} = \frac{2}{2} = 1$$

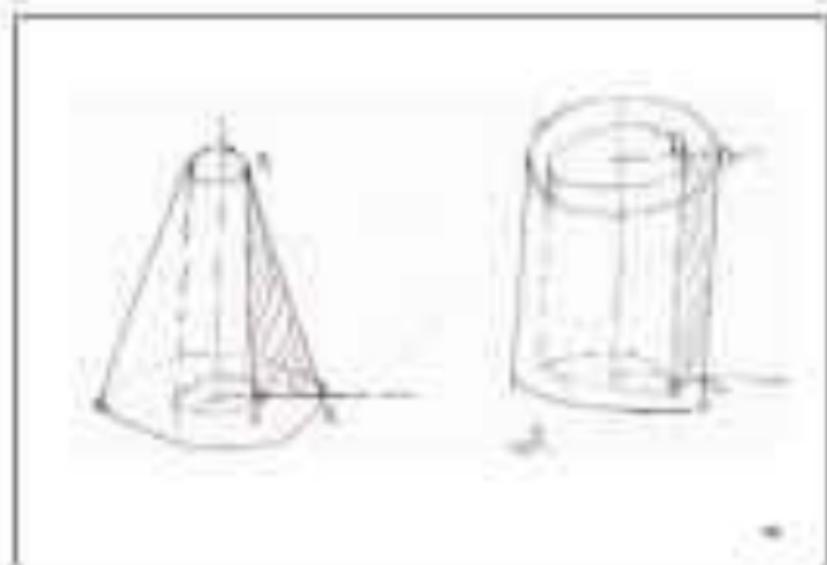
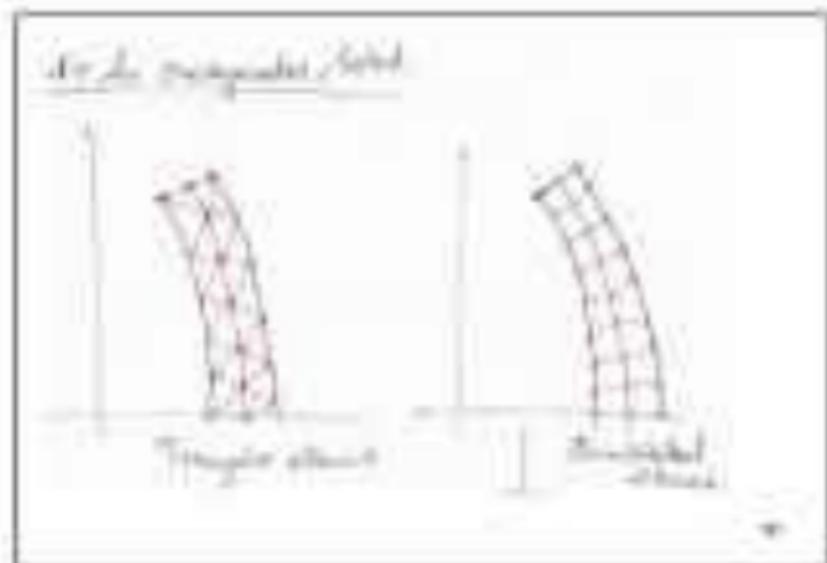
Q3) Find the value of $\frac{dy}{dx}$ at $x=1$ by

$$f_3 = \frac{(1+2x^2) - 1}{2} = \frac{2}{2} = 1$$

Q4) Find the value of

$$f_4 = \frac{(1+2x^2) - 1}{2} + \frac{(1+2x^2) - 1}{2} = 1 + 1 = 2$$





In 2D displacement across thickness assumed to be zero. Other displacements (say u and v) are functions of x and y .

Similarly, in the case of axisymmetric solid, the circumferential displacement (in θ) is zero (i.e. $w=0$) and have u and v displacements are as the functions of r and z respectively.

$$\begin{aligned}
 u &= u(r, z) \\
 v &= v(r, z) \\
 w &= 0
 \end{aligned}$$



Definition of shape function for axisymmetric triangular element



Consider an axisymmetric triangular element with nodes 1, 2 and 3 as in the figure.



Let $u_1, u_2, u_3, u_4,$ and u_5 be nodal displacements

$$5 \text{ d.o.f.} \quad \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$$


It should have 3 adjacent, shared face-to-face elements to provide for compatibility for displacement function at any point inside the element

$$\begin{aligned} u(x, y) &= \sum_{i=1}^3 N_i u_i \\ N_1(x, y) &= \frac{1}{3} (1 - x) (1 - y) \\ N_2(x, y) &= \frac{1}{3} x (1 - y) \\ N_3(x, y) &= \frac{1}{3} x y \end{aligned} \quad \leftarrow \text{??}$$

Asymptotic analysis

$$\begin{aligned} \text{if } (x, y) \text{ inside and } u(x, y) &= \\ N_1(x, y) &= \frac{1}{3} (1 - x) (1 - y) \\ N_2(x, y) &= \frac{1}{3} x (1 - y) \\ N_3(x, y) &= \frac{1}{3} x y \end{aligned}$$

Nodal displacements

$$U_1 = U_2 = U_3 = U_4 = 0$$

$$U_5 = U_6 = U_7 = U_8 = U_9$$

$$U_{10} = U_{11} = U_{12} = U_{13}$$

and

$$U_4 = U_5 = U_6 = U_7 = U_8 = U_9$$

$$U_{10} = U_{11} = U_{12} = U_{13}$$

$$U_{14} = U_{15} = U_{16} = U_{17}$$



Eqn 2 in matrix form

$$\begin{matrix} \text{Eqn 1a} \\ \left\{ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} \right\} \end{matrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \left\{ \begin{matrix} U_1 \\ U_2 \\ U_3 \end{matrix} \right\}$$

$$\left\{ \begin{matrix} P_4 \\ P_5 \\ P_6 \end{matrix} \right\} = \begin{bmatrix} k_4 & 0 & 0 \\ 0 & k_5 & 0 \\ 0 & 0 & k_6 \end{bmatrix} \left\{ \begin{matrix} U_4 \\ U_5 \\ U_6 \end{matrix} \right\}$$

$$20 \times 10^8 = 10$$

$$K_{11} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}$$

$$= \frac{1}{20} [10 \times 10^8 \times 1, 10 \times 10^8 \times 0, 10 \times 10^8 \times 0]$$

$$= \frac{10 \times 10^8}{20} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \times 10^7 & 0 & 0 \\ 0 & 5 \times 10^7 & 0 \\ 0 & 0 & 5 \times 10^7 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix}$$

$$2. \text{ Now } K_{12} = K_{21} \quad \text{--- (1)}$$

$$K_{12} = K_{21} = K_{13} = K_{31} \quad \text{--- (2)}$$

When starts to compute length along

$$K_{12} = \frac{EA}{L} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for } L = 20$$

Assuming the shape function N_i is constant along the element, the displacement u can be written as $u = N_i d_i$. The strain ϵ is given by $\epsilon = \frac{du}{dx} = \frac{dN_i}{dx} d_i$. The stress σ is given by $\sigma = E \epsilon = E \frac{dN_i}{dx} d_i$. The element stiffness matrix k is given by $k = \int_{x_1}^{x_2} B^T D B dx$, where $B = \frac{dN_i}{dx}$ and $D = E$.

For a 2D element, the shape functions are $N_1 = \frac{x_2 - x}{x_2 - x_1}$, $N_2 = \frac{x - x_1}{x_2 - x_1}$. The stiffness matrix k is given by $k = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

For a 3D element, the shape functions are $N_1 = \frac{(x_2 - x)(y_2 - y)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_2 = \frac{(x - x_1)(y_2 - y)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_3 = \frac{(x - x_1)(y - y_1)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_4 = \frac{(x_2 - x)(y - y_1)(z - z_1)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$. The stiffness matrix k is given by $k = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

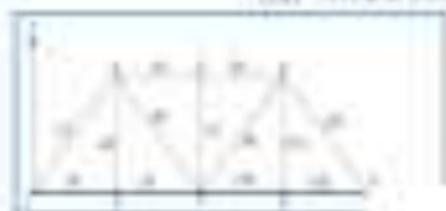
For a 4-noded element, the shape functions are $N_1 = \frac{(x_2 - x)(y_2 - y)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_2 = \frac{(x - x_1)(y_2 - y)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_3 = \frac{(x - x_1)(y - y_1)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_4 = \frac{(x_2 - x)(y - y_1)(z - z_1)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$. The stiffness matrix k is given by $k = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

For a 8-noded element, the shape functions are $N_1 = \frac{(x_2 - x)(y_2 - y)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_2 = \frac{(x - x_1)(y_2 - y)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_3 = \frac{(x - x_1)(y - y_1)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_4 = \frac{(x_2 - x)(y - y_1)(z - z_1)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_5 = \frac{(x_2 - x)(y_2 - y)(z - z_1)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_6 = \frac{(x - x_1)(y_2 - y)(z - z_1)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_7 = \frac{(x - x_1)(y - y_1)(z - z_1)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$, $N_8 = \frac{(x_2 - x)(y - y_1)(z_2 - z)}{(x_2 - x_1)(y_2 - y_1)(z_2 - z_1)}$. The stiffness matrix k is given by $k = \frac{EA}{L} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Miscellaneous Items

COORDINATE SYSTEMS

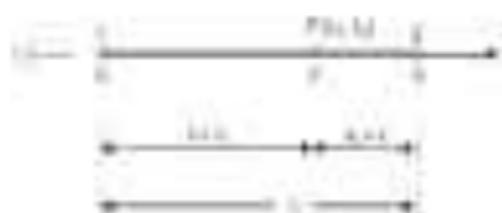
- (i) Global coordinates
- (ii) Local coordinates and
- (iii) Natural coordinates.


Natural Coordinates

- A natural coordinate system is a coordinate system which permits the specification of a point within the element by a set of dimensionless numbers, whose magnitude never exceeds unity.
- It is obtained by assigning weights to the nodal coordinates, so defining the coordinates of any point inside the element.
- The use of natural coordinate system is advantageous in assembling element properties (stiffness matrices), since closed form integrations formulas are available when the integrations are in terms of coordinate system.

Natural Coordinates in one Dimension:

Consider the one loaded bar element shown in Fig. Let the natural coordinates of point P be ξ , ξ_1 , ξ_2 and the Cartesian coordinates be x . Node 1 and node 2 have the Cartesian coordinates x_1 and x_2 .



Since natural coordinates are nothing but weights to the nodal coordinates, total weights at any point is unity i.e., $L_1 + L_2 = 1$ and also

$$L_1 + L_2 = 1$$

$$L_1 x_1 + L_2 x_2 = x$$

In matrix form

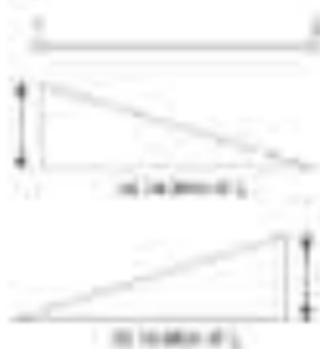
$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 \\ x_1 & x_2 \end{Bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\
 &= \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \\
 &= \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x_1 \\ -x_1 + x_2 \end{Bmatrix}
 \end{aligned}$$

Therefore, L_1 and L_2 are the shape of the element

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \frac{x_2 - x}{x_2 - x_1} \\ \frac{x - x_1}{x_2 - x_1} \end{Bmatrix}$$

The variation of L_1 and L_2



The number of nodal force integration over unity length is

$$\int_0^1 L_1^p L_2^q dx = \frac{1^{p+q+1}}{(p+q+1)}$$

Integrate the following over the unity length 1 of the element

$$\text{[We take } p=2, q=3]$$

$$\text{(i) } \int_0^1 L_1^2 dx$$

$$\int_0^1 L_1^2 dx = \frac{2 \times 1}{2+1+1}$$

$$= \frac{1}{2+1} = \frac{1}{3} \quad \text{Answer}$$

$$\text{(ii) } \int_0^1 L_1^2 L_2 dx = \frac{1 \times 1}{(2+1+1)}$$

$$= \frac{1 \times 1 \times 1}{1 \times 4 \times 1 \times 2} = \frac{1}{8} \quad \text{Answer}$$

Natural Coordinates in Three Dimensions



Tetrahedron coordinates

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} \quad \xi_i = \frac{V_i}{V} \quad (0 \leq \xi_i \leq 1)$$

$$\int_{\text{tetrahedron}} f(x,y,z) dV = \frac{V}{6} \int_{\text{tetrahedron}} f(\xi_1, \xi_2, \xi_3, \xi_4) d\xi_1 d\xi_2 d\xi_3$$

f(x,y,z)

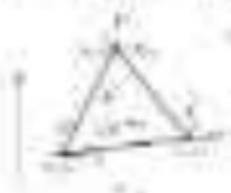
tetrahedron coordinates

$$\int_{\text{tetrahedron}} f(x,y,z) dV = \frac{V}{6} \int_{\text{tetrahedron}} f(\xi_1, \xi_2, \xi_3, \xi_4) d\xi_1 d\xi_2 d\xi_3$$

$$\frac{V}{6} \int_{\text{tetrahedron}} f(\xi_1, \xi_2, \xi_3, \xi_4) d\xi_1 d\xi_2 d\xi_3$$

Constant Strain Triangle (CST) or C0

This is the simplest element in terms of shape & number of nodes & degrees of freedom.



Constant Strain Triangle



Linear Displacement Field

Applications of the CST Element

- Used in cases where the stress gradient is small
- Used in applications where element is under uniform stress.
- First step in finite element analysis to other types of elements for structural analysis (e.g. plate and shell elements)
- The element has only one unknown (i.e. constant) of 2D problem.

Linear Strain Triangle (LST) or C1

This element is used where gradient of displacement is needed.



Linear Strain Triangle

Shape Functions

$$\begin{aligned}
 N_1 &= \frac{1}{3}(1-x-y) \\
 N_2 &= \frac{1}{3}(1+x) \\
 N_3 &= \frac{1}{3}(1+y) \\
 N_4 &= \frac{1}{3}x \\
 N_5 &= \frac{1}{3}y \\
 N_6 &= \frac{1}{3}xy
 \end{aligned}$$



Linear Displacement Field

Global (Nodal) Degrees of Freedom


- Notes:**
- Global DOFs are used to describe the whole body
 - Global DOFs are used to describe the whole body
 - Global DOFs are used to describe the whole body

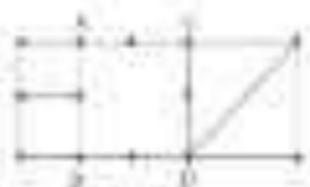
Notes

- Global DOFs are used to describe the whole body
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- Global DOFs are used to describe the whole body



Weakness of element property:

- Don't have translational gaps in the structure in 2D models



Element connected to each other at node D

Quality

→ more regular geometry is desirable as it is easier to implement in software codes.

Types of element:

- 1D element: beam, truss, bar, cable
- 2D element: plate, shell
- 3D element: solid
- Fluid flow element

•

Application of the element property

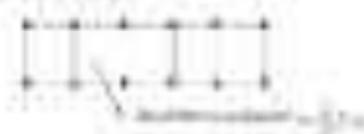
- Applied for use in the finite element method (FEM) and other engineering software
- Used in structural analysis
- Used in fluid flow analysis
- Used in heat transfer analysis

→ More regular geometry is desirable as it is easier to implement in software codes. It is also important to ensure that the element is not too distorted (e.g. too thin or too thick) to avoid numerical issues.

Source of Finite Element Solution

- FE Model - 4 nodal beam with 2DOF per node (2D)
- 2D structure - where nodes are the joints
- nodes at positions are defined nodes of 2D.
- FE Model - structure is broken into elements (2D)

→ displacement at each node is assumed to be the value of flexibility matrix at node.



assuming 2D

- FE Model is built from the structure
- Element displacement field is built a matrix into flexibility matrix



These 2DOF solution of the beam corresponds to the rigid body motion.

So 2DOF solution corresponds to rigid motion (2D)
 Because the displacement is fixed (P=0)

Special Case
Types of Cases

- Modeling Error (Mesh, Joints ... Material)
 - Discretization Error (Mesh, parameters ...)
 - Numerical Error (舍入误差 / Rounding Error)
- Large differences in stiffness of different parts in FE model may cause ill-conditioning in FE equations. These groups correlate with large errors.
 - Ill-conditioned system of equations can lead to huge changes in solution with small changes in input (High level of sensitivity)

Characteristics of FE Solution

Inherent error in FE solution. Inherent error is the error which cannot be eliminated by any number of discretization. Error of the solution due to the ill-conditioned system of equations. Error due to FE discretization is function of discretization.

Types of Discretization

- Geometrical** - when the mesh is discretized. It is called as the spatial discretization.
- Material** - when the material is discretized. It is called as the material discretization.
- Parameter** - when the parameters are discretized. It is called as the parameter discretization.
- Modeling** - when the model is discretized.
- Computational** - Truncation of the FE eqs. and numerical solution.

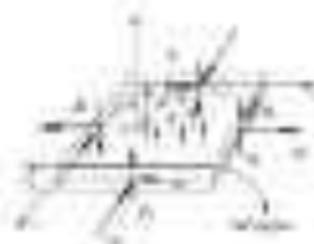
Plate and Shell Elements

Plate Theory - 1/2 dim

- Kirchhoff
- Reissner-Mindlin

Applications

- Aircraft
- Turbines
- Ships



Generalized Behavior of a Plate

Two-Dimensional Beam-Element Theory

(assumes similar to those of the beam theory)

It straightens along the normal to the end surface remains straight and normal to the surface and remains undeformed. And so there is no transverse shear deformation.

$$d_1 = d_2 = 0$$



$$\begin{bmatrix}
 u_1 \\
 u_2 \\
 u_3 \\
 u_4
 \end{bmatrix}$$

Thin Plate Theory (Mindlin Plate Theory)

It is the extension of a plate model (BKT) e.g. 1.1, 2.1, 3.1, 4.1 (1D) to a three-dimensional description of the plate. Thus the Kirch plate theory by Mindlin should be applied. This theory accounts for the angle change within a cross-section. That is,

$$\gamma_x \neq 0, \quad \gamma_z \neq 0$$

This means that a face which is normal to the mid surface before the deformation will not be so after the deformation.


Thin Plate
Normal to the Plate

perpendicular



(a)

$$\gamma_x = \alpha - \frac{\partial y}{\partial x}$$

Skewed Plate Element

perpendicular to the mid surface

perpendicular



(b)

perpendicular



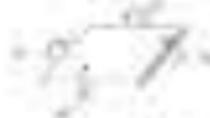
Shell and Solid Elements

Shell: Membrane and bending behavior



- Design
- No. of DOF (DOF) at each node depends on the element
 - Commonly 6 DOF
 - 3 DOF in-plane
 - 3 DOF, including the out-of-plane DOF

Free surface: Shell structures - bending behavior
 of plate (shell type)



- Shell type
- Thin shell type
 - Thick shell type

All types of elements have 8 nodes
 (8 nodes are used for all elements)

Typical 1-D Solid Elements

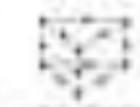
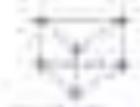
Linear shape



Quadratic shape



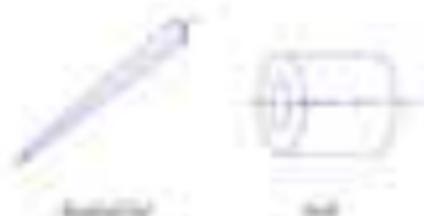
Free



Recall using the tables (10 slide) introduction showed in 1-21
 stress analysis (Buckingham's) that it is 100% the physical problem

a 1D (axially) & one material (EM)

both of the two components below



Axial Rod

To determine Governing Equation,

1. Equilibrium equation: $\frac{d\sigma_x}{dx} + B_x = 0$
2. strain-displacement relation: $\epsilon_x = \frac{du}{dx}$
3. Constitutive relation: $\sigma_x = E\epsilon_x$

Using the above relations, we have

$$\begin{aligned}
 \sigma_x &= E \frac{du}{dx} \\
 \frac{d}{dx} \left(E \frac{du}{dx} \right) + B_x &= 0
 \end{aligned}$$

$$D_1 - \text{occurs} \Rightarrow \frac{d^2 u}{dx^2} + q_1 = 0$$

If q_1 is constant as per unit length, then

$$\frac{d^2 u}{dx^2} = -q_1$$

$$\Rightarrow \int \frac{d^2 u}{dx^2} = \frac{d^2 u}{dx^2} = -q_1$$

$$\int \left(\frac{d^2 u}{dx^2} + q_1 \right) dx = 0 \quad \text{is the governing equation}$$

Or with Boundary Conditions, the Boundary Value Problem

will be solved:

in FE, the above equation takes the form

$\mathbf{T}^T \mathbf{u} = \mathbf{r}$ and inserted in Finite Element Equation

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Weighted Residual Method

$$\int_{\Omega} \left(\frac{d^2 u}{dx^2} + q \right) dx = 0, \quad \forall \Omega \subset \Omega$$

$$\forall \Omega \subset \Omega \subset \Omega$$

$$\text{For } \int_{\Omega} \left(\frac{d^2 u}{dx^2} + q \right) dx = 0, \quad \text{Theorem 10}$$

We want to minimize the weighted residual in order to develop
 the Galerkin method

$$\int_{\Omega} w(x) dx = 0$$

$$\Rightarrow \int_{\Omega} w \left(\frac{d^2 u}{dx^2} + q \right) dx = 0$$

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$$= \int_{\Omega_1} N^T A \frac{d^2 u}{dx^2} dx + \int_{\Omega_1} N^T p dx$$

This is known as **strong form** for the first element.

$$= \left(N^T A \frac{d^2 u}{dx^2} \right)_{x_1} - \int_{x_1}^{x_2} \frac{dN^T}{dx} dx \frac{du}{dx} dx + \int_{\Omega_1} N^T p dx$$

This is known as **weak form** for the first element.

$$= N^T(x_1) u'_1 - N^T(x_2) u'_2 - \int_{x_1}^{x_2} \frac{dN^T}{dx} dx \frac{du}{dx} dx + \int_{\Omega_1} N^T p dx$$

$$= \frac{dN^T}{dx} = \mathbf{B} \quad \text{and here is the matrix}$$

Solving, piecewise and expressing in $\mathbf{U}(\mathbf{x})$

$$u(\mathbf{x}) = \mathbf{N}(\mathbf{x}) \mathbf{u}_n + \mathbf{R}_n(\mathbf{x})$$

and numbering \mathbf{U} as u_1, \dots, u_n , it is a statement, we get:

$$\mathbf{U} = \int_{\Omega_1} \frac{dN^T}{dx} \mathbf{A} \left(\frac{dN^T}{dx} \mathbf{u}_n + \frac{dR_n}{dx} \right) dx + \int_{\Omega_1} N^T p dx$$

$$\mathbf{U} = \int_{\Omega_1} \frac{dN^T}{dx} \mathbf{A} \left(\frac{dN^T}{dx} \mathbf{u}_n + \frac{dR_n}{dx} \right) dx + \int_{\Omega_1} N^T p dx$$

$$\mathbf{U} = \begin{bmatrix} \int_{\Omega_1} N_1^T A N_1 dx & \int_{\Omega_1} N_1^T A N_2 dx \\ \int_{\Omega_1} N_2^T A N_1 dx & \int_{\Omega_1} N_2^T A N_2 dx \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \int_{\Omega_1} N_1^T p dx \\ \int_{\Omega_1} N_2^T p dx \end{bmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = - \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$Q = -kU + P$$

Thus, we obtain the discrete governing equation.

$$P = kU$$

References

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All the best ...

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