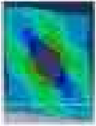
Lecture Notes on

Finite Element Methods for Aerospace



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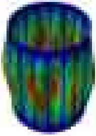

AMRITA | School of Engineering

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Prepared by
Dr. [illegible]

This work is dedicated to all my teachers who have imparted their knowledge and guidance throughout this course

Course Objective

Understand the concepts of mathematical modeling of engineering problems by introducing the Finite Element Methods and to help the students use this method and commercial software package to solve complex aerospace structures.

Course Outcome

- [illegible]
- [illegible]
- [illegible]
- [illegible]
- [illegible]

Syllabus

Unit 1 [illegible]

Unit 2 [illegible]

Unit 3 [illegible]

Introduction

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Where the Course Fits

The field of Mechanics can be subdivided into 3 major areas:

Mechanics
- Theoretical
- Applied
- **Computational**

Computational Mechanics

Branches of Computational Mechanics can be distinguished according to the physical focus of attention:

Computational Mechanics
- Nano and Micro mechanics
- Continuum Mechanics: Solids and Structures, Fluids, Multiphysics
- Systems

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Computational Solid and Structural Mechanics

A taxonomic subdivision of problems in Computational Solid and Structural Mechanics (CSSM) is

Computational Solid and Structural Mechanics (CSSM) { Statics / Dynamics

[illegible]

CSSM Statics

A further subdivision of problems in CSSM Statics is

CSSM Statics { Linear / Nonlinear

CSM Linear Statics

For the numerical simulations on the computer the [illegible] choice is opened after evaluation method:

CSM Linear Statics:
- *Finite Element Method*
- *Finite Difference Method*
- *Boundary Element Method*
- *Finite Volume Method*
- *Spectral Method*
- *Mesh-Free Method*

The finite difference method (FDM) is a numerical technique that solves differential equations by approximating their derivatives with difference equations.

The boundary element method (BEM) is a numerical technique used to solve linear partial differential equations. [illegible] to solve problems in elasticity.

The finite volume method (FVM) is a discretization method that approximates PDEs by dividing the domain into control volumes and evaluating the integral of the function [illegible].

The spectral element method (SEM) is a numerical technique for solving partial differential equations that combines the flexibility of finite element methods with the accuracy of spectral methods.

[illegible] Linear Statics by FEM

Having selected the FEM for discretization, we must next pick a *formulation* and a *solution method*:

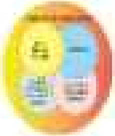
Formulation of FEM Model
- Displacement
- Equilibrium
- Mixed
- Hybrid

Solution of FEM Model
- Stiffness
- Flexibility
- Mixed

Field of Mechanics :

- Theoretical
- Applied
- Computational
- Experimental

Computational mechanics is strongly interdisciplinary.



Methods of analysis (solution) of differential equations

- Analytical methods
  - Exact methods (e.g., separation of variables and Laplace transformation methods)
  - Approximate methods (e.g., Rayleigh-Ritz and Galerkin methods)
- Numerical methods
  - Numerical solution of differential equations
    - Numerical integration
    - Finite differences
  - Finite or discrete element method

Theoretical Mechanics

Deals with fundamental laws and principles of mechanics studied for their intrinsic (basic characteristics) value.

Applied Mechanics

Transforms this theoretical knowledge to scientific and engineering applications by construction of mathematical models of physical phenomena.

Computational Mechanics

Solves specific problems by combining mathematical models with numerical methods implemented in digital computers. (Simulation)

Experimental Mechanics

Puts physical laws, mathematical models and numerical simulations under stringent test of observation.

Most of the Solid Mechanics Problems are Boundary Value Problems (BVP).

Any solutions to this problem must satisfy the following conditions.

Field Condition: Field variables, displacements (strains), stresses (stress resultants) must satisfy the governing equations (GE).

GE : Differential Equations (DE) or Variational Formulations (VF) and

Boundary Condition (BC) – Kinematic and static

Differential Equations:

Governing equations (GE) derived by vectorial treatment of the principles of mechanics

GEs also obtained using Euler-Lagrangian equations

GE for beam (fourth order):

$$EI\left(\frac{d^4w}{dx^4}\right)=p$$

GE for plate (isotropic):

$$\left(\frac{d^4w}{dx^4}\right)+2\left(\frac{d^4w}{dx^2dy^2}\right)+\left(\frac{d^4w}{dy^4}\right)=\frac{p}{D}$$

where,

$$D=\frac{Eh^3}{12(1-\mu^2)}$$

(1) What is the Finite Element Method?

The Finite Element Method (FEM) is a sophisticated **numerical scheme** to interpolate the **approximate solution** to common **boundary-value problems** of engineering and mathematical physics.

- The finite element method is a numerical method for solving problems of engineering and mathematical physics.
- Useful for problems with complicated geometries, loadings, and material properties where analytical solutions can not be obtained.

The Finite element method is a numerical method for solving problems of engineering and mathematical physics.

Engineering Problems:

Structural analysis
Heat transfer
Fluid flow
Mass transport and
Electromagnetic potential

- Advantages over analytical mathematical solutions are problems involving complicated geometries, loadings and material properties used in FEM method.
- Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities at any location in a body and are thus valid for an infinite number of locations in the body.
- Analytical solutions typically involve solving ordinary or partial differential equations. However due to complex geometries, varying load conditions, and diverse material properties, these solutions are often unattainable.Hence we need to rely on numerical methods, such as finite element method for acceptable solutions.

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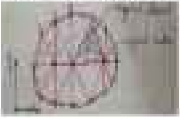
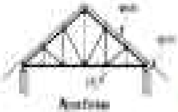
Discretization:

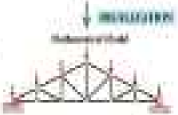
Modelling a body by dividing it into an equivalent system of smaller bodies(finite elements) interconnected at points common to two or more elements (nodes) and boundary lines is called discretization.

The solution for structural problems typically refers to determining the displacements/stress at each node and the stresses within each element making up the structure that is subjected to applied loads.

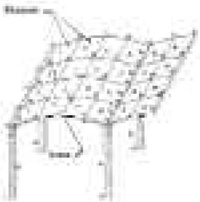


- Model the body by dividing it into an equivalent system of smaller bodies (finite elements) interconnected at points common to two or more elements (nodes) and/or boundary lines and/or surfaces.

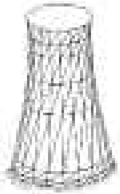




FE MODELS

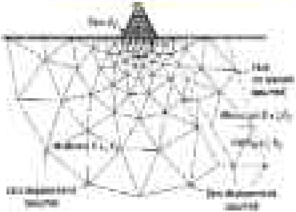


Shell Roof

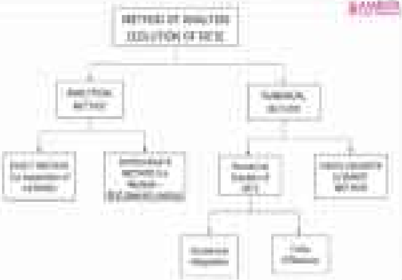


Cooling Tower

Plane stress analysis of Buttress Dam



FE MODELS



Non-Structural Problems: The nodal unknown maybe temperature or fluid pressures due to thermal or fluid fluxes.

General Steps of FEM:

Two general approaches associated with the FEM:

1) Force or Flexibility Method.

2) Displacement or Stiffness Method.

Flexibility Method:

It is used to internal forces as the unknown of the problem. To obtain the governing equations, first the equilibrium equations are used. Then, the necessary additional equations are found by introducing compatibility equations. The result is a set of algebraic equations for determining the redundant and unknown forces.

Stiffness Method:

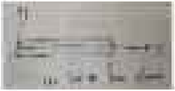
Assumes the displacement of the nodes as the unknowns of the problem. Compatibility conditions requires that elements connected at a common node, along common edge or on a common surface before loading remains connected at the node, edge/surface after deformation takes place are initially satisfied.

Then, the governing equations are expressed in terms of nodal displacements using the equations of equilibrium and an applicable law relating forces to displacements.

FINITE ELEMENT METHOD

1. Discretize and select the element types

- The choice between 1D, 2D, or 3D elements in finite element method (FEM) analysis is a critical task for designers, as selecting the appropriate element type significantly affects the accuracy and efficiency of the simulation.



Rotating a part if loading and geometry should be axisymmetric.
Axisymmetric (ax) triangular element through 360 (deg)

FINITE ELEMENT METHOD

2. Select the (displacement) function

- Involves choosing a displacement function within each element.
- The function is defined within the element using the nodal values of the element.
- Linear, quadratic, and cubic polynomials are frequently used.
- Trigonometric series can also be used for a 2D element the displacement function is a function of the coordinates in its plane.

Linear Quadratic Cubic

3. Define the strain/displacement and s/c relationships

- Strain/displacement and s/c relationships are necessary for deriving the equations for each finite element. In 1-D deformation are having relation:

$\epsilon_x = du/dx$ for small strains

- In addition stress must be related to the strain through the law called Constitutive law.
- The ability to define the material behavior accurately is most important in obtaining acceptable results. $\sigma_x = E\epsilon_x$

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1.Deriving the element stiffness matrix and equations:

➢ Direct equilibrium method:

- Stiffness matrix & and element equations relating nodal forces to nodal displacement are obtained using force equilibrium conditions for basic element, along with force/deformation relationships.

- This method is necessary/most easy for 1D elements [K] generation. Spring, bar, and beam elements are essential components in structural analysis.

➢ Work energy method:

- This method is useful to 2D and 3D elements for developing [K] matrix and equations.

- Principles of virtual work (virtual displacements), the principle of minimum potential energy and of Castigliano's theorem are used frequently used to derivation of element equations.

- Principle of virtual work applicable for any material behavior, principle of minimum potential energy and Castigliano theorem are applicable only to elastic materials.

- Functional methods analogous to be used with the principle of minimum potential energy and also useful in obtaining [K] matrix and equations for heat transfer, fluid flow problems.

- Let Π – Functional

- f(x,y) denote a function f of two variables x and y.
- We have Π=Π f(x,y).
  - Where Π is a function of the function 'f'

$$[k] = \iiint [B] [D] [B]^T \, dv$$

- Method of weighted residuals (WR):
- Useful for developing the element equations; popular is Galerkin's method.
- These methods yield the same results as the energy methods wherever the energy methods are applicable.
- The WR are especially useful when a functional such as potential energy is not readily available.
- The behavior of an element can be described by equations obtained through any of the methods.

$$\{f\} = [k]\{d\}$$

Where {f} – Vector of element nodal forces

[k] – Element stiffness matrix

{d} – Vector of unknowns (or) displacement degrees of freedom

FINITE ELEMENT METHOD

5. Assemble the element equations to obtain the global equations and introduce boundary conditions:

- The individual element equations can be added together using a method of superposition (direct stiffness method) to get global equilibrium equations

$\{F\} = [K]\{d\}$

$\{F\}$: Global nodal force vector

$[K]$: Global stiffness

$\{d\}$: unknown DoF

$[K]$: (singular matrix because its determinant is equal to zero)

- To remove this singularity problem, we must invoke boundary conditions so that the structure remains in place instead of moving as a rigid body.

FINITE ELEMENT METHOD

6. Solve for the unknown DoF:

- Set of simultaneous algebraic equations ...

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7. Solve for the element ε and σ:

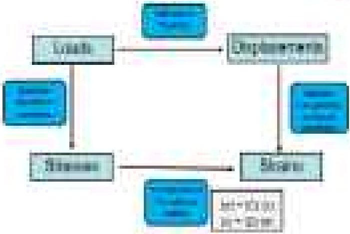
- [illegible]

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \ldots \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \ldots$$

8. Interpret the results:

- Finally interpret and analyze the results for use in the design/analysis process [illegible]

- Post processors computer programs helps to interpret the results.



Brief history of FEM

Authors	Year	Contribution
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Various steps involved in the finite element analysis are:

(i) Select suitable field variables and the elements.

(ii) Discritise the continua.

(iii) Select interpolation functions.

(iv) Find the element properties.

(v) Assemble element properties to get global properties.

(vi) Impose the boundary conditions.

(vii) Solve the system equations to get the nodal unknowns.

(viii) Make the additional calculations to get the required values.

$$[k]_e \{\delta\}_e = \{F\}_e$$

$$[k]\{\delta\} = \{F\}$$

where $[k]$ is stiffness matrix

$\{\delta\}$ is displacement vector and

$\{F\}$ is force vector in the coordinate directions

Advantages

- Irregular Boundaries
- General Loads
- Different Materials
- Boundary Conditions
- Variable Element Size
- Easy Modification
- Nonlinear Problems (Geometric, Material & Boundary)

Objectives of This FEM Course

- Understand the fundamental ideas of the FEM
- Know the behavior and usage of each type of elements covered in this course
- Be able to prepare a suitable FE model for given problems
- Can interpret and evaluate the quality of the results (know the physics of the problems)
- Be aware of the limitations of the FEM (don't misuse the FEM - a numerical tool)

Computer Implementations

- Preprocessing (build FE model, loads and constraints)
- FEA solver (assemble and solve the system of equations)
- Postprocessing (sort and display the results)

Available Commercial FEM Software Packages

- ANSYS (General purpose, PC and workstations)
- SDRC/I-DEAS (Complete CAD/CAM/CAE package)
- NASTRAN (General purpose FEA on mainframes)
- ABAQUS (Nonlinear and dynamic analyses)
- COSMOS (General purpose FEA)
- ALGOR (PC and workstations)
- PATRAN (Pre/Post Processor)
- HyperMesh (Pre/Post Processor)
- Dyna-3D (Crash/impact analysis)

Advantages of General Purpose Programs

- Easy input - preprocessor.
- Solves many types of problems
- Modular design - fluids, dynamics, heat, etc.
- Can run on PC's now.
- Relatively low cost.

Disadvantages of General Purpose Programs

- High development costs.
- Less efficient than smaller programs.
- Often proprietary. User access to code limited.

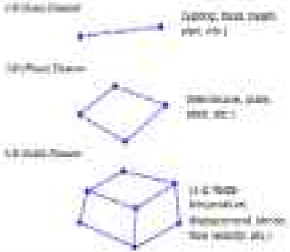
Feature

- Obtain a set of algebraic equations to solve for unknown nodal quantity (displacement).
- Secondary quantities (stresses and strains) are expressed in terms of nodal values of primary quantity.

General Description of the Finite Element Method

- Discretization of the structure
- Selection of the proper interpolation or displacement function
- Derivation of element stiffness matrices and load vectors
- Assemblage of element equations to obtain the overall equilibrium equations
- Solution for the unknown nodal displacements
- Computation of element strains and stresses

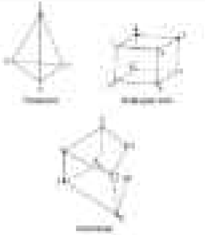
Types of Finite Elements



BASIC ELEMENT SHAPES

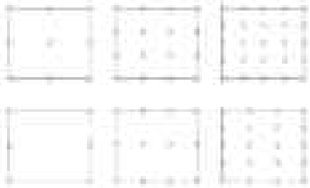
One-Dimensional Elements

Two Dimensional Elements

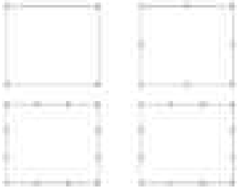


Three Dimensional Finite Elements

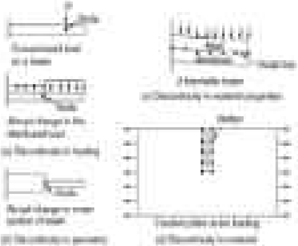
Lagrange family of rectangular elements



Serendipity family rectangular elements



Location of Nodes



Number of Elements

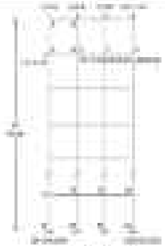
NODE NUMBERING SCHEME

- The FE analysis involves matrix equations in which the matrices involved will be banded.

- Large FE problems possible due to banded nature of the matrices.

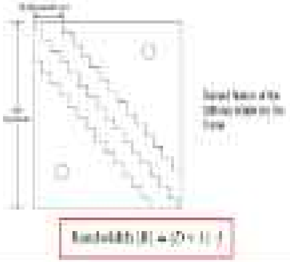
Most of matrices are symmetric, computer storage can be substantially reduced by storing only the elements involved in half bandwidth instead of storing the entire matrix.

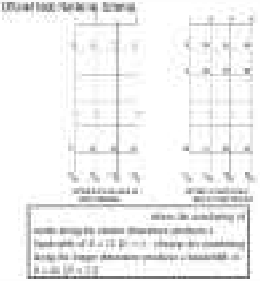
- The bandwidth of the global characteristic matrix depends on the node numbering scheme and the number of dof considered per node.

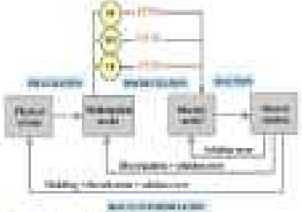


A Three-bay Frame

As an example, consider a three-bay frame with rigid joints, 20 stories high. Assuming that there are 3 dof per node, there are 252 unknowns in the final equations (including the dof corresponding to the fixed nodes), and if the entire stiffness matrix is stored in the computer, it will require 252^2 = 63,504 locations. The bandwidth (strictly speaking, half bandwidth) of the overall stiffness matrix can be shown to be 15, and thus the storage required for the upper half band is only 15 × 252 = 3780 locations.







The three stages of computer-based simulation:

Idealization, discretization and solution

Strong Form (SF):

Presented as a system of ordinary or partial differential equations in space and/or time, complemented by appropriate boundary conditions. Occasionally this form may reduce to algebraic equations.

Weak Form (WF):

Presented as a weighted integral equation that relaxes the strong form into a domain-averaging statement.

Variational Form (VF):

Presented as a functional whose stationary conditions generate the weak and strong forms.

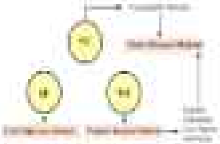
Variational calculus is a set of rules and techniques by which one can pass from one form to another.

Why Weak and Variational Forms?

The following accrue may be offered:

1. Basis for important discrete methods of approximation, notably FEM
2. Characterize overall quantities of interest to engineers
3. Clarify treatment of boundary & interface conditions
4. Permit unified mathematical treatment of questions of existence, stability, errors, etc. Also provide guidelines as to how to achieve desirable behavior in discrete models

Mathematical Model Forms as Sources of Discretization Methods



Strong, Weak and Variational forms as sources of numerical approximation methods

Derivation of Element Matrices and Vectors

The characteristic matrices and vectors of finite element can be derived by using any of the following approach:

1. Direct Approach

   In this method, direct physical reasoning is used to establish the element properties in terms of pertinent variables.
   Applicability to simple types of elements.

2. Variational Approach

   The FE analysis is interpreted as an approximate means for solving variational problems.
   Most physical and engineering problems can be formulated in variational form, the FEM can be readily applied for finding their approx. solutions.
   Major limitation – requires the physical and engineering problem to be stated in variational form, which may not be possible in all cases.

3. Weighted Residual Approach

   The element matrices and vectors are derived directly from the governing differential equations.
   Can be applied to almost all the practical problems of science and engineering. Different Procedures: Galerkin method, Least squares method.

- If the physical problem is stated as a variational problem, and the exact solution, which minimizes the integral I, can not be found easily, we would like to find an approximate solution that approximately minimizes the integral I.
- Similarly, if the problem is stated in terms of differential equations and B.C. and if the correct solution, which satisfies all the equations exactly, can not be obtained easily, we would like to find an approx. solution that satisfies the B.C. exactly but not governing differential equations.

Variational Formulation

Functional: stationary value of the functional is sought.

The governing function for SM: Total potential energy or Total complementary energy

Using the calculus of variation, stationary value of a functional obtained by equating first variation of the functional to zero

$$\delta \Pi = 0$$

The solution to the field variable is one that yield stationary value for the PE functional(Π)

Also should satisfy all the prescribed boundary BC's

Boundary Conditions

1. Geometric BC (displacements, slope, derivatives of slope) and
2. Static or Force BC (forces, moments)

Vertical deflection v at any point taken as field variable, it must be satisfy the BC, which is in an equilibrium condition. $EI\frac{d^4v}{dx^4} = q$

Geometric BC x = 0

Deflection v = 0, slope dv/dx = 0

Static or Force BC x = L

Moment $EI\frac{d^2v}{dx^2} = 0$

Shear $EI\frac{d^3v}{dx^3} = 0$

Potential Energy

Potential energy is the capacity of the forces to do work. For structural systems or deformable bodies the forces are classified as external forces due to loads and internal forces due to stresses (or stress resultants). Thus the total potential energy is the sum of the potential energy of the external and internal forces.

A spring system

For the spring shown in Fig. , as the deformation in the spring changes from 0 to u, the strain energy of the spring is $\frac{1}{2}ku^2$. Thus the total potential energy of the model is

$$\Pi = \frac{1}{2}ku^2 - Pu \qquad (3.8)$$

Let u, v and w be the field displacement components. Following the concept used in the example, the total potential energy of the external forces can be expressed by

$$W = -\iiint_V (f_x u + f_y v + f_z w)\, dV - \iint_{S_1} (T_x u + T_y v + T_z w)\, dS \qquad (2.46)$$

$$(W_p = -W)$$

where W_p is the external work done by the external and body forces.

The Eq.2.46 is given in matrix form as

$$W = -\iiint_V \{u\}^T \{f\}\, dV - \iint_{S_1} \{u\}^T \{T\}\, dS$$

If concentrated loads P_i are present on the body or structure, the potential energy due to these loads have to be included as,

$$W = -\iiint_V \{u\}^T \{f\}\, dV - \iint_{S_1} \{u\}^T \{T\}\, dS - \sum_{i=1}^{n} P_i u_i \qquad (2.48)$$

where u_i is the displacement corresponding to P_i and the summation extends all such loads.

Next, we would derive an expression for the internal strain energy for a three-dimensional solid subjected to stress system. Consider a differential element shown in Fig.2.6 subjected to a stress of σ_x.

The differential or incremental work done due to increase in displacement is

$$dW = \sigma_x \, dy \, dz \left(u + \frac{\partial u}{\partial x} dx \right) [illegible]$$

$$= \sigma_x \left(\frac{\partial u}{\partial x} \right) dx\,dy\,dz \qquad [illegible]$$

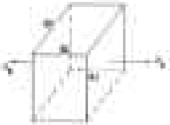


Fig. [illegible] Three Dimensional differential element

[illegible]

Since, $\frac{\partial u}{\partial x} = \varepsilon_x$ we get,

$$dW = \sigma_x \, d\varepsilon_x \, dx\,dy\,dz \qquad [illegible]$$

As the strain increases from zero to the final value ε_x, the differential work done which is the strain energy developed in the element is

$$\int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \, dV$$

[illegible]

If the element is subjected to all the six stress components, the total strain energy developed in the element is given by

$$U = \left[\int_0^{\varepsilon_x} \sigma_x d\varepsilon_x + \int_0^{\varepsilon_y} \sigma_y d\varepsilon_y + \int_0^{\varepsilon_z} \sigma_z d\varepsilon_z + \int_0^{\gamma_{xy}} \tau_{xy} d\gamma_{xy} + \int_0^{\gamma_{yz}} \tau_{yz} d\gamma_{yz} + \int_0^{\gamma_{zx}} \tau_{zx} d\gamma_{zx}\right] dV$$

The integration is performed from initial to the final state of strain.

Let U_0 be the strain energy density, i.e., strain energy per unit volume. Then generalizing Eq.3.18 we can express the increment in strain energy density as

$$dU_0 = \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} \quad (3.20)$$

From differential calculus we know that dU_0 can be expressed as

$$dU_0 = \frac{\partial U_0}{\partial \varepsilon_x} d\varepsilon_x + \frac{\partial U_0}{\partial \varepsilon_y} d\varepsilon_y + \frac{\partial U_0}{\partial \varepsilon_z} d\varepsilon_z + \frac{\partial U_0}{\partial \gamma_{xy}} d\gamma_{xy} + \frac{\partial U_0}{\partial \gamma_{yz}} d\gamma_{yz} + \frac{\partial U_0}{\partial \gamma_{zx}} d\gamma_{zx} \quad (3.21)$$

Comparing Eqs. 2.52 and 2.51, we get

$$\frac{\partial U_o}{\partial \varepsilon_x} = \sigma_x, \quad \frac{\partial U_o}{\partial \varepsilon_y} = \sigma_y, \quad \frac{\partial U_o}{\partial \gamma_{zx}} = \tau_{zx} \qquad (2.53)$$

Thus, the strain energy density has the property that its partial derivative with regard to any strain component is the corresponding stress component.

Eq 2.53 can be expressed in matrix form as,

$$\left\{\frac{\partial U_o}{\partial \varepsilon}\right\} = \{\sigma\} \qquad (2.54)$$

Substituting $\{\sigma\} = [C]\{\varepsilon\}$ from Eq. 2.12 into Eq 2.54, we get

$$\left\{\frac{\partial U_o}{\partial \varepsilon}\right\} = [C]\{\varepsilon\} \qquad (2.55)$$

Integration of Eq 2.55 with respect to the strains yields

$$U_o = \frac{1}{2}\{\varepsilon\}^T[C]\{\varepsilon\}$$

$$\text{or} \quad U_o = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} \qquad (2.56)$$

In the case of linearly elastic isotropic body, substituting for [C] from Eq 2.1[illegible], assuming T = 0 and simplifying we get,

$$U_o = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) +$$

$$\frac{(1+\nu)}{E}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)$$

[illegible])

$$U_0 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz})$$

The total strain energy U, can be obtained by integrating U_0 over the volume of the solid. Thus,

$$U = \int\int\int_V U_0\, dV \qquad ([illegible])$$

And substituting for U_0 from Eq. [illegible],

$$U = \frac{1}{2}\int\int\int_V \{\varepsilon\}^T[D]\{\varepsilon\}\, dV$$

[illegible]

In case initial stress and strain, and temperature effects are to be considered, the stress-strain relation given by Eq. [illegible] is to be used for the evaluation of strain energy. It can be shown

$$\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\} \qquad ([illegible])$$

$$U = \int\int\int_V \left(\frac{1}{2}\{\varepsilon\}^T[D]\{\varepsilon\} - \{\varepsilon\}^T[D]\{\varepsilon_0\} + \{\varepsilon\}^T\{\sigma_0\}\right) dV$$

In the above equation, a term $\frac{1}{2}\int\int\int_V \{\varepsilon_0\}^T[D]\{\varepsilon_0\}\, dV$ is omitted and since it is a constant and has no effect in the variational process to get the equilibrium equation.

[illegible]

Then the total potential energy Π is defined as the sum of the external potential energy and strain energy

$$\Pi = \Omega + U$$

Substituting from Eqs.2.47 and 2.48,

$$\pi = \frac{1}{2}\iiint_V \{\varepsilon\}^T[C]\{\varepsilon\}\,dV - \iiint_V \{u\}^T\{X\}\,dV - \iint_{S_1} \{u\}^T\{q\}\,dS$$

[illegible]

Substituting for $\{u\}$ and $\{\varepsilon\}$: $\{u\} = [N]\{\delta\}$ $\{\varepsilon\} = [B]\{\delta\}$

$$\pi_e = \frac{1}{2}\iiint_V \{\delta\}^T[B]^T[C][B]\{\delta\}\,dV - \iiint_V \{\delta\}^T[N]^T\{X\}\,dV - \iint_{S_1} \{\delta\}^T[N]^T\{q\}\,dS$$

where V is the volume of the element and S_1 is the portion of the boundary where forces are prescribed.

[illegible]

According to the principle of stationary potential energy, the first variation of Π must be zero for equilibrium condition. Thus taking the first variation [illegible] we get

$$[illegible] = 0$$

$$[illegible] = 0$$

$$[K]\{q\} = \{Q\}$$

47

In case the element is subjected to initial strains and stresses, the strain energy Π,

$$[illegible] = 0$$

Thus the equilibrium equation is reduced to the same form.

$$[K]\{q\} = \{Q\}$$

where

$$[K] = [illegible]$$

$$\{Q\} = [illegible]$$

48

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

$$[illegible] \quad (2.43)$$

[illegible]

$$[illegible] \quad (2.44)$$

[illegible]

[illegible]

$$[illegible] \quad (2.45)$$

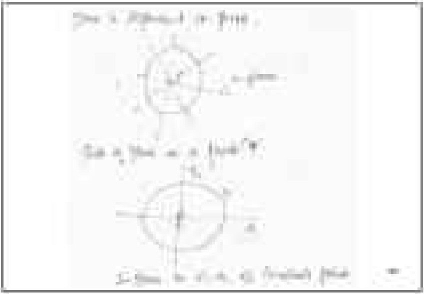
[illegible]

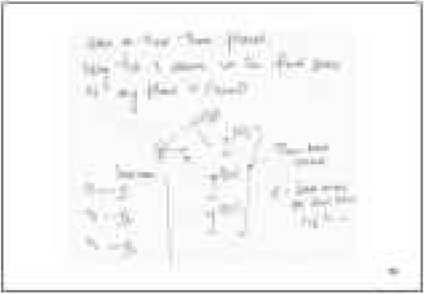
$$[illegible] \quad (2.46)$$

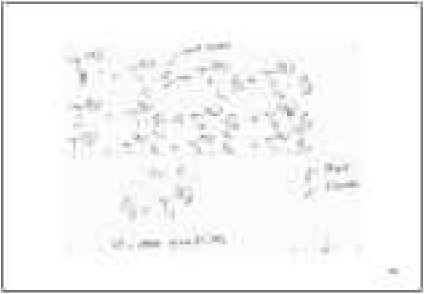
[illegible]

[illegible]

State of Stress at a point







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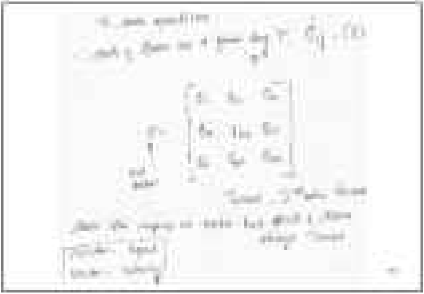
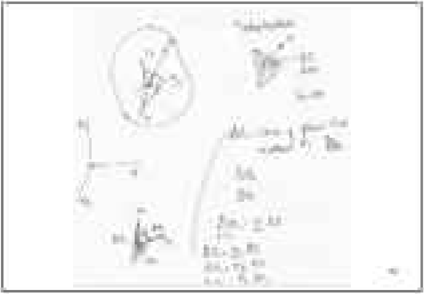
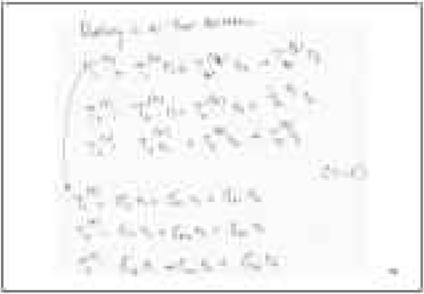







Indicial Notation

[illegible]

[illegible]

Linear Constitutive Equations

Linear Constitutive Equations

The constitutive law expresses the relationship among stresses and strains. In theory of elasticity, usually it is considered as linear.

one dimensional stress analysis, the linear constitutive law is stress is proportional to
strain and the constant of proportionality is called Young's modulus.
It is very well known as Hooke's law: stress is proportional to strain and the constant of proportionality is called Young's modulus.

Finite Element for 3-D Problems

Solid Elements for 3-D Problems

Hooke's Law:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}$$

Equilibrium Equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$$

Isotropic:

An isotropic body has material properties that are the same in every direction at a point in the body, i.e., the properties are independent of orientation at a point in the body.

Orthotropic:

An orthotropic body has material properties that are different in three mutually perpendicular directions at a point in the body and, further, has three mutually perpendicular planes of material property symmetry. Thus, the properties depend on orientation at a point in the body.

Anisotropic:

An anisotropic body has material properties that are different in all directions at a point in the body. No planes of material property symmetry exist. Again, the properties depend on orientation at a point in the body.

The most general relation is expressed among the six components of stresses and strains and is called "**Generalized Hooke's Law**". This may be stated as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

[illegible]

where D is 6×6 matrix of constants of elasticity to be determined by experimental investigations for each material. As D is symmetric matrix $[D] = [D]^T$, there are 21 material properties for linear elastic **Anisotropic Materials**.

[illegible]

If a material exhibits symmetry with respect to planes within the body. Such materials are called **Orthotropic materials**. Hence for orthotropic materials, the number of constants reduces to 9 as shown below

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 \\ & & D_{33} & 0 & 0 & 0 \\ & Sym & & D_{44} & 0 & 0 \\ & & & & D_{55} & 0 \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

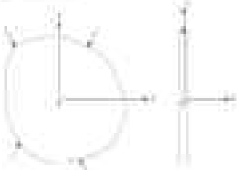
strain stress relation for isotropic material is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{2(1+\nu)}{E} & 0 & 0 \\ & & & & \frac{2(1+\nu)}{E} & 0 \\ & & & & & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Plane Stress Problems

The thin plates subject to forces in their plane only, fall under this category of the problems.



$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\sigma_z = \mu\varepsilon_x + \mu\varepsilon_y + (1-\mu)\varepsilon_z = 0$$

$$\varepsilon_z = -\frac{\mu}{1-\mu}\left(\varepsilon_x + \varepsilon_y\right)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

[illegible]

[illegible]

Plane Strain Problems

A long body subject to significant lateral forces but very little longitudinal forces falls under this category of problems.

(a)

(b)

[illegible]

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$\gamma_{xz} = \gamma_{yz} = 0$ means τ_{xz} and τ_{yz} are zero.

$\varepsilon_z = 0$ means

$$\varepsilon_z = \frac{\sigma_z}{E} - \mu\frac{(\sigma_x + \sigma_y)}{E} = 0$$

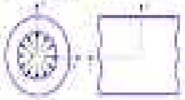
$$\sigma_z = \mu(\sigma_x + \sigma_y)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

- Plane strain:

$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \qquad (\sigma_z \neq 0)$

A long structure with a uniform cross section and transverse loading along its length (z direction).



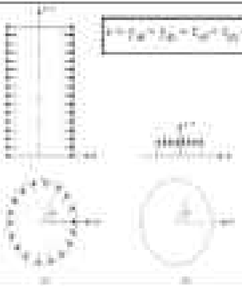
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \tau_{xy0} \end{Bmatrix}$$

in the plane strain case.

Axi-Symmetric Problems

Axi-symmetric structures are those which can be generated by rotating a line or curve about an axis. Cylinders are the common examples of axisymmetric structures. If such structures are subjected to axisymmetric loadings like radial or internal or external pressure, such as self weight or live load uniform over the surface.

$$\gamma_{r\theta} = \gamma_{\theta z} = \tau_{r\theta} = \tau_{\theta z} = 0$$



Hence there are only four stress components. The strain displacement relations for these components are:

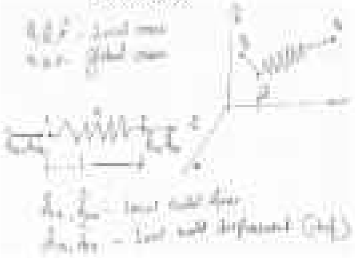
$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \text{ and}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ & 1-\mu & \mu & 0 \\ & & 1-\mu & 0 \\ & & & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix}$$

Stiffness Method

Stiffness Method



[illegible]

[illegible]

[illegible]

In matrix form

$$u = \left[1 - \frac{x}{L} \quad \frac{x}{L} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \text{—— (2)}$$

or

$$u = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where $N_1 = 1 - \frac{x}{L}$ and $N_2 = \frac{x}{L}$

N_1 and N_2 = the shape functions

N_1 [illegible]

N_2 [illegible]

[illegible]
[illegible]

[illegible]

[illegible]

[illegible]

where (8) in matrix:

$$\begin{Bmatrix} f_{1} \\ f_{2} \end{Bmatrix}^{e} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_{1} \\ u_{2} \end{Bmatrix}^{e}$$

$$\{f\} = [k]\{u\} \quad \text{— stiffness equation}$$

$$k = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

↑ stiffness matrix

(5) Assemble the element stiffness equations to obtain the global structure and introduce B.C.

$$K = [K] = \sum_{e=1}^{N} k^{(e)}$$

$$F = \{F\} = \sum_{e=1}^{N} f^{(e)}$$

[illegible]

Element 1: $\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{2x} \end{Bmatrix}$ ---(1)

Element 2: $\begin{Bmatrix} f_{2x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_{2x} \\ u_{3x} \end{Bmatrix}$ ---(2)

Element 1 and 2 must remain connected at common node & displacement.
This is called continuity or compatibility requirement.

$$u_{2x}^{(1)} = u_{2x}^{(2)} = u_{2x}$$

$F = K d$

F — global nodal forces

d — global nodal displacements

$$K = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix}$$ global stiffness matrix

Assembling the total stiffness matrix by superposition

We have eqn (1) and (2)

$$k^{(1)} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{matrix} d_{1x} \\ d_{3x} \end{matrix} \qquad k^{(2)} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{matrix} d_{3x} \\ d_{2x} \end{matrix}$$

Consider d_{1x} d_{2x} d_{3x}

$$k_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ [illegible] \\ f_{3x}^{(1)} \end{Bmatrix} \quad \text{--- (3)}$$

[illegible]

From equilibrium at every node

Solve eqs ①–②–③

Simplify above

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

[illegible]

$[K]$

Properties of [K] matrix

- It is symmetric.
- It is singular [illegible] rigid body [illegible]
- The main diagonal terms are always positive [illegible]

B.C. $d = 0$, $d = 0$ and apply loadings

$$\begin{Bmatrix} 0 \\ -1000 \end{Bmatrix} = \begin{bmatrix} 5000 & -2000 \\ -2178 & 5700 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{4x} \end{Bmatrix}$$

$d = 0.91$ mm

$d = 1.166$ mm

[illegible]

$$\begin{Bmatrix} f \\ f \\ f \\ f \end{Bmatrix} = \begin{bmatrix} [illegible] \end{bmatrix} \begin{Bmatrix} d \\ d \\ d \\ d \end{Bmatrix}$$

[illegible]

[illegible]

[illegible]

[illegible]

Element [illegible]

[illegible]

Element [illegible]

[illegible]

Tensile force [illegible]

Element [illegible]

[illegible]

Compressive force

[illegible]

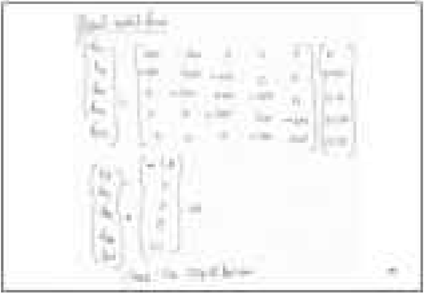
Apply BC [illegible]

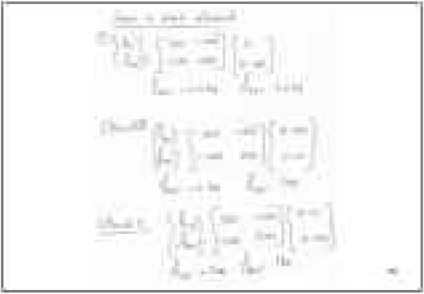
[illegible]

[illegible]

Simplify and rearrange

[illegible]





Element 1:

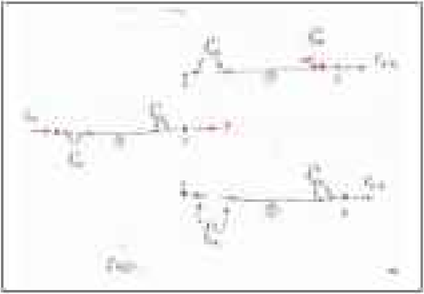
Compatibility Conditions at node

Compatibility Condition at node 2

[illegible]

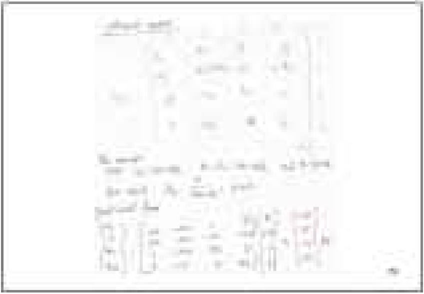
For nodal equilibrium condition

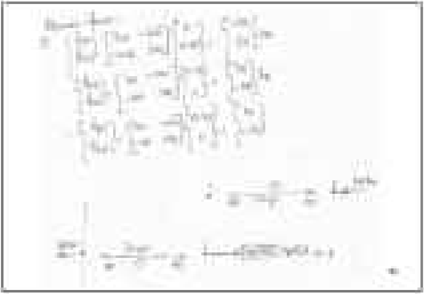
[illegible] — (1)



[illegible]

[illegible]





Spring Element

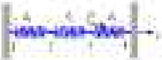
[illegible]

[illegible]

Properties of stiffness matrix

- [illegible]
- [illegible]
- [illegible]
- [illegible]
- [illegible]
- [illegible]

Example



Given: For the spring system shown above,

[illegible]

Find: (a) the global stiffness matrix

(b) displacements of nodes 2 and 3

(c) the reaction forces at nodes 1 and 4

(d) the force in the spring 2

[illegible]

$$k_1 = \begin{bmatrix} [illegible] & [illegible] \\ [illegible] & [illegible] \end{bmatrix} \text{[illegible]} \quad [illegible]$$

$$k_2 = \begin{bmatrix} [illegible] & [illegible] \\ [illegible] & [illegible] \end{bmatrix} \text{[illegible]} \quad [illegible]$$

$$k_3 = \begin{bmatrix} [illegible] & [illegible] \\ [illegible] & [illegible] \end{bmatrix} \text{[illegible]} \quad [illegible]$$

[illegible]

$$K = [illegible]$$

[illegible]

[illegible] **[illegible]**

[illegible]

$$[illegible] \quad [illegible]$$

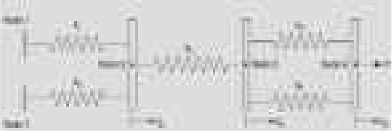
$$KQ = F$$

[illegible]

$$[illegible] \quad [illegible]$$

[illegible]

$$[illegible] \quad [illegible]$$



Solution:

[illegible]

[illegible]

$$[k^{(1)}] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$ (corresponding degrees of freedom [illegible])

$$[k^{(2)}] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$ (corresponding degrees of freedom [illegible])

[illegible]

$$[k^{(3)}] = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$ (corresponding degrees of freedom [illegible])

$$[k^{(4)}] = \begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}$$ (corresponding degrees of freedom [illegible])

$$[k^{(5)}] = \begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix}$$ (corresponding degrees of freedom [illegible])

[illegible]

[illegible]

$$
[K] = \begin{bmatrix} k_1+k_2 & -(k_1+k_2) & 0 & 0 \\ -(k_1+k_2) & k_1+k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4+k_5 & -(k_4+k_5) \\ 0 & 0 & -(k_4+k_5) & k_4+k_5 \end{bmatrix}
\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}
$$

(columns: u_1, u_2, u_3, u_4)

[illegible]

[illegible]

$$
\overline{F} = \begin{Bmatrix} F_1 \\ 0 \\ 0 \\ P \end{Bmatrix}, \quad \overline{U} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}
$$

[illegible]

[illegible]

$$[K]\{U\} = \{F\}$$

[illegible]

[illegible matrix equation]

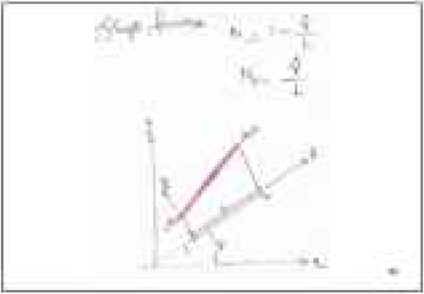
[illegible]

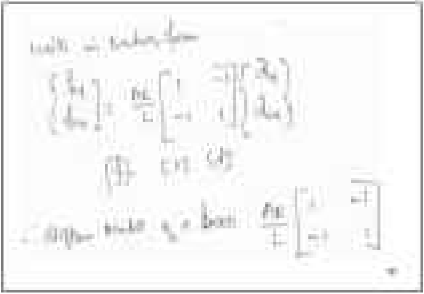
[illegible matrix equation]

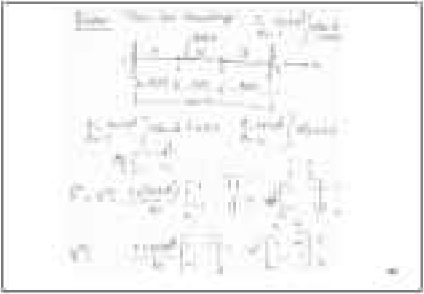
[illegible]

Axial Element

BAR ELEMENT (Linear Elastic Bar)

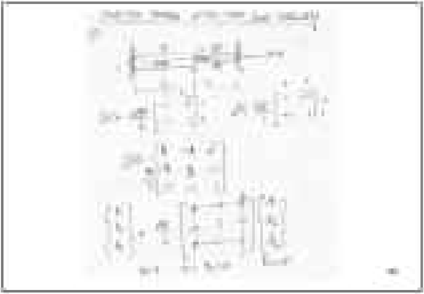






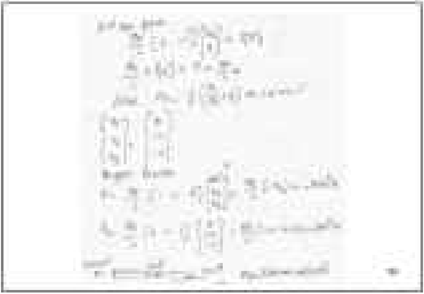
[illegible]

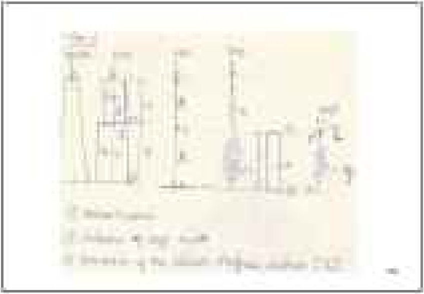
[illegible]

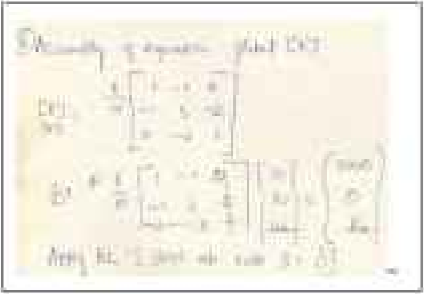


[illegible]

[illegible]







[illegible]

i. Computation of the elemental strains and stress from nodal displacement

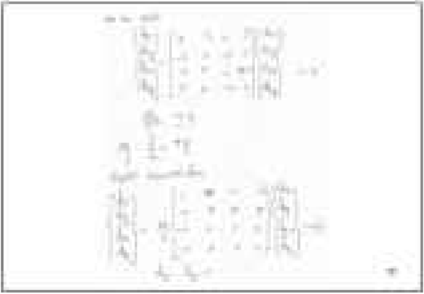
[illegible]

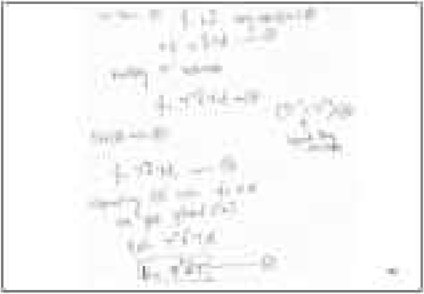
Truss Analysis

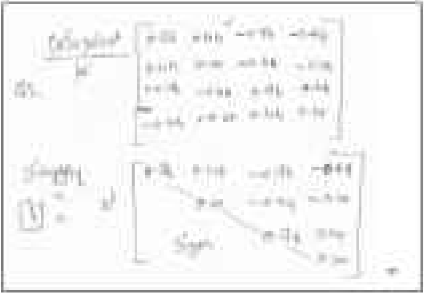
Example:

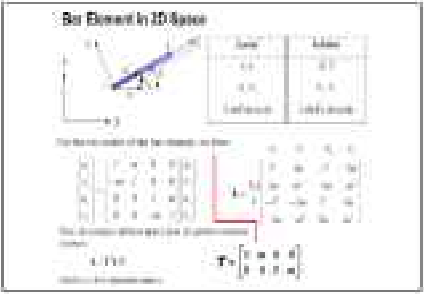
[illegible]

[illegible]





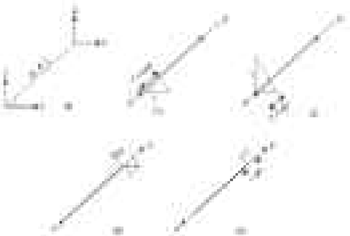



Bar Element in 2D Space

Truss Element

$$\delta = \frac{PL}{AE} \qquad \therefore P = \frac{EA}{L}\delta$$

$$\therefore \text{If } \delta = 1, \; P = \frac{EA}{L}$$



(i) Unit displacement of node 1 in x-direction

$$P = \frac{EA}{L}\cos\theta$$

$$k_{11} = P\cos\theta = \frac{EA}{L}\cos^2\theta$$

$$k_{21} = P\sin\theta = \frac{EA}{L}\cos\theta\sin\theta$$

$$k_{31} = -P\cos\theta = -\frac{EA}{L}\cos^2\theta$$

$$k_{41} = -P\sin\theta = -\frac{EA}{L}\cos\theta\sin\theta$$

(ii) Unit displacement in coordinate direction 2

$$k_{12} = P\cos\theta = \frac{EA}{L}\sin\theta\cos\theta$$

$$k_{22} = P\sin\theta = \frac{EA}{L}\sin^2\theta$$

$$k_{32} = -P\cos\theta = -\frac{EA}{L}\sin\theta\cos\theta$$

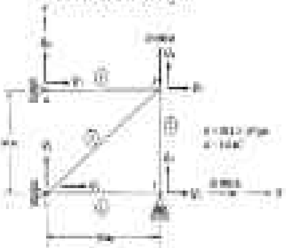
$$k_{42} = -P\sin\theta = -\frac{EA}{L}\sin^2\theta$$

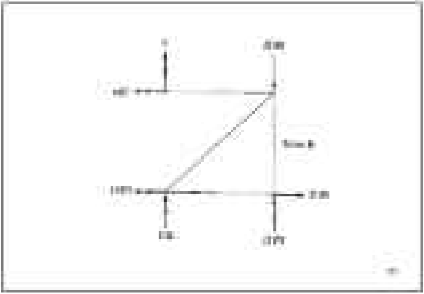
∴ The stiffness matrix is

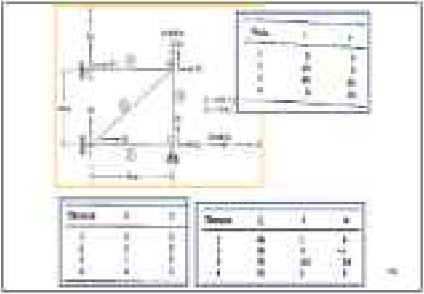
$$[k] = \frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$[k] = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Plane Truss Analysis







Element Stiffness Matrix

$$\mathbf{k} = \frac{E_e A_e}{\ell_e}\begin{bmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{bmatrix}$$

$\ell = \cos\theta \qquad m = \cos\phi$

Global dof: 1 2 3 4

$$\mathbf{k}^1 = \frac{29.5\times10^6}{40}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix}1\\2\\3\\4\end{matrix}$$

Columns: 5 6 3 4

$$\mathbf{k}^2 = \frac{29.5\times10^6}{30}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{matrix}5\\6\\3\\4\end{matrix}$$

1 2 5 6

$$
\mathbf{k}^{2} = \frac{29.5 \times 10^{6}}{50}
\begin{bmatrix}
.64 & .48 & -.64 & -.48 \\
.48 & .36 & -.48 & -.36 \\
-.64 & -.48 & .64 & .48 \\
-.48 & -.36 & .48 & .36
\end{bmatrix}
\begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}
\qquad (\text{columns } 1, 2, 5, 6)
$$

$$
\mathbf{k}^{3} = \frac{29.5 \times 10^{6}}{40}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{matrix} 7 \\ 8 \\ 5 \\ 6 \end{matrix}
\qquad (\text{columns } 7, 8, 5, 6)
$$

$$
\mathbf{K} = \frac{29.5 \times 10^{6}}{600}
\begin{bmatrix}
22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\
5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\
-15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\
-7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\
-5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\
0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}
\qquad (\text{columns } 1 \ldots 8)
$$

$$\frac{29.5\times10^6}{600}\begin{bmatrix}15 & 0 & 0\\ 0 & 22.68 & 5.76\\ 0 & 5.76 & 24.32\end{bmatrix}\begin{Bmatrix}Q_3\\ Q_5\\ Q_6\end{Bmatrix}=\begin{Bmatrix}20\,000\\ 0\\ -25\,000\end{Bmatrix}$$

Solution of these equations yields the displacements

$$\begin{Bmatrix}Q_3\\ Q_5\\ Q_6\end{Bmatrix}=\begin{Bmatrix}27.12\times10^{-3}\\ 5.65\times10^{-3}\\ -22.25\times10^{-3}\end{Bmatrix}\text{in}$$

The nodal displacement vector for the entire structure can therefore be written as

$$Q=[0,0,27.12\times10^{-3},0,5.65\times10^{-3},-22.25\times10^{-3},0,0]^T\ \text{in}$$

$$\sigma=\frac{E_e}{l_e}\begin{bmatrix}-l & -m & l & m\end{bmatrix}q$$

$$\sigma_1=\frac{29.5\times10^6}{40}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 27.12\times10^{-3}\\ 0\end{Bmatrix}$$

$= 20\,000$ psi

The stress in member 2 is given by

$$\sigma_2=\frac{29.5\times10^6}{50}\begin{bmatrix}0.8 & 0.6 & -0.8 & -0.6\end{bmatrix}\begin{Bmatrix}5.65\times10^{-3}\\ -22.25\times10^{-3}\\ 27.12\times10^{-3}\\ 0\end{Bmatrix}$$

$= -21\,880$ psi

$\sigma_3 = -5208$ psi

$\sigma_4 = 4167$ psi

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_5 \\ R_6 \end{Bmatrix} = [illegible]$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_5 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 7136.0 \\ 20375.0 \\ -4167.0 \\ 0 \end{Bmatrix} \text{N}$$

Assume the area of cross section of all the members is the same. E = [illegible] $= 10^6$ N/mm². Find the force in the vertical member.

$A = 100$ mm²

$$[k] = \frac{EA}{L}\begin{bmatrix} C_x^2 & C_xC_y & -C_x^2 & -C_xC_y \\ C_xC_y & C_y^2 & -C_xC_y & -C_y^2 \\ -C_x^2 & -C_xC_y & C_x^2 & C_xC_y \\ -C_xC_y & -C_y^2 & C_xC_y & C_y^2 \end{bmatrix}$$

$C_x = 0.5, \quad C_y = 0.866$

$$[k] = \frac{EA}{[illegible]}\begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix}$$

Member [illegible]

[illegible]

$$[k] = \frac{EA}{[illegible]}\begin{bmatrix} 1.44 & 2.5 & -1.44 & -2.5 \\ 2.5 & 4.33 & -2.5 & -4.33 \\ -1.44 & -2.5 & 1.44 & 2.5 \\ -2.5 & -4.33 & 2.5 & 4.33 \end{bmatrix}$$

Member 1 i(0, 0), j(8660, 5000), L = 10000

$C_x = 0.866, C_y = 0.5$ $\theta = 30$

$$\begin{matrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 3 & 4 \end{matrix}$$

$$[k_1] = \frac{EA}{10000}\begin{bmatrix} 7.5 & 4.33 & -7.5 & -4.33 \\ 4.33 & 2.5 & -4.33 & -2.5 \\ -7.5 & -4.33 & 7.5 & 4.33 \\ -4.33 & -2.5 & 4.33 & 2.5 \end{bmatrix} \begin{matrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 2 \end{matrix}$$

Member 2 i(8660, 5000), j(8660, 0);

L = 5000, $C_x = 0, C_y = -1.0$ $\theta = 270$

$$\begin{matrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 3 & 4 \end{matrix}$$

$$[k_2] = \frac{EA}{5000}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix} \begin{matrix} 0 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \end{matrix}$$

Member 4 i(86.6, 50.0), j(173.2, 0);
$L = 100\text{cm},\ C_x = 0.866, C_y = -0.5$ $\quad \theta = 330$

$$
\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix}
$$

$$
[k_4] = \frac{EA}{1000}\begin{bmatrix} 7.5 & -4.33 & -7.5 & 4.33 \\ -4.33 & 2.5 & 4.33 & -2.5 \\ -7.5 & 4.33 & 7.5 & -4.33 \\ 4.33 & -2.5 & -4.33 & 2.5 \end{bmatrix} \begin{matrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \end{matrix}
$$

Member 5 i(86.6, 150.0), j(173.2, 0);
$L = 173.2\text{cm}, C_x = 0.5, C_y = -0.866$ $\quad \theta = 300$

$$
\begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{matrix}
$$

$$
\frac{EA}{1000}\begin{bmatrix} 1.44 & -2.50 & -1.44 & 2.50 \\ -2.50 & 4.33 & 2.50 & -4.33 \\ -1.44 & 2.50 & 1.44 & -2.50 \\ 2.50 & -4.33 & -2.50 & 4.33 \end{bmatrix} \begin{matrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 0 \end{matrix}
$$

the stiffness matrix for the structure is given by,

$$[K] = \frac{EA}{1000}\begin{bmatrix} 18.66 & -10.0 \\ -10.0 & 15.0 \end{bmatrix}$$

The nodal load vector is

$$\{P\} = \begin{Bmatrix} 100.0 \\ 50.0 \end{Bmatrix}$$

Hence, the equilibrium equation is,

$$\frac{EA}{1000}\begin{bmatrix} 18.66 & -10.0 \\ -10.0 & 15.0 \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} 100.0 \\ 50.0 \end{Bmatrix}$$

Solving the above equations we get,

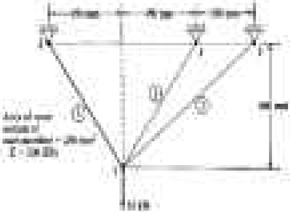
$$r_1 = \frac{0.5373}{A}$$

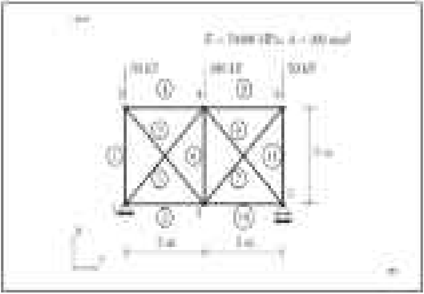
$$r_2 = \frac{0.5358}{A}$$

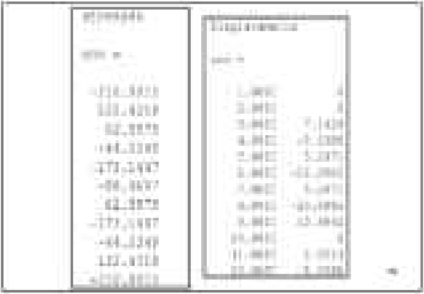
$$\{F\} = [k]\{d\}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix}$$

Class Exercise




5 m
5 m

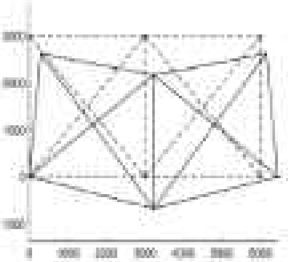


```
reactions

ans =

   1.0e+05 *

    0.0000    0.0000
    0.0000    1.0000
    0.0001    1.0000
```



Displacements

$Q_1 = 0.0265117\text{ cm}$

$Q_6 = 0.0004[illegible]\text{ cm}$

$Q_7 = 0.347903\text{ cm}$

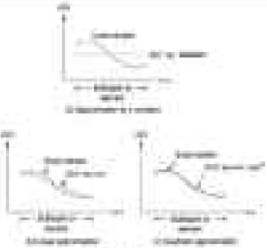
$Q_8 = -0.3600[illegible]\text{ cm}$

THREE-DIMENSIONAL TRUSSES

$$\mathbf{k} = \frac{E_e A_e}{\ell_e}\begin{bmatrix} \ell^2 & \ell m & \ell n & -\ell^2 & -\ell m & -\ell n \\ \ell m & m^2 & mn & -\ell m & -m^2 & -mn \\ \ell n & mn & n^2 & -\ell n & -mn & -n^2 \\ -\ell^2 & -\ell m & -\ell n & \ell^2 & \ell m & \ell n \\ -\ell m & -m^2 & -mn & \ell m & m^2 & mn \\ -\ell n & -mn & -n^2 & \ell n & mn & n^2 \end{bmatrix}$$

Formulation of
Shape/Interpolation Function

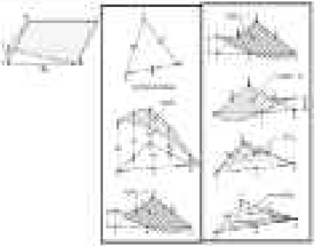
Interpolation Models



Polynomial approximation in One Dimension

- Interpolation polynomial of order one – linear element
- Interpolation polynomial of order two – higher order element

- If P is fixed, the discretization of the region can be improved by two methods:
  - r-method: the location of the nodes are altered without changing the total number of elements.
  - h-method: number of elements is/are increased.
- To improve the accuracy of problem, increase the order of polynomial, called p-method



Guidelines to selecting approximate functions for displacements

1. Polynomials are commonly used approximate functions. Polynomials are simple and give linear/quadratic/cubic … variation of displacement.

2. The approximate function should be continuous within the element. Therefore the triangular shape function yields continuous values of 'u' within the element and prevents openings, overlaps and jumps because of the continuous and smooth variation in u.

$$d_{2x}^{(1)} = d_{2x}^{(2)}$$

Prepared by
Dr. [illegible]

4. The approximate function should allow for rigid body displacement and for a state of constant strain within the element.

When a function satisfies the above conditions,

the function is called Complete function of that element

This completeness condition is necessary for convergence to the exact answer.

This also that the interpolation function must allow for [illegible] means that the function must capable of yielding constant values.

[illegible]

[illegible]

POLYNOMIAL FORM OF INTERPOLATION FUNCTIONS

If a polynomial type of variation is assumed for the field variable $\phi(x)$ in a one-dimensional element, $\phi(x)$ can be expressed as

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \cdots + \alpha_m x^n$$

Similarly, in two- and three-dimensional finite elements the polynomial form of interpolation functions can be expressed as

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \cdots + \alpha_m y^n$$

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} zx + \cdots + \alpha_m z^n$$

For $n = 1$ (linear model)

One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x$$

Two-dimensional case:

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

Three-dimensional case:

$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

For $n = 2$ (quadratic model)

One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

Two-dimensional case:

$$\phi(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy$$

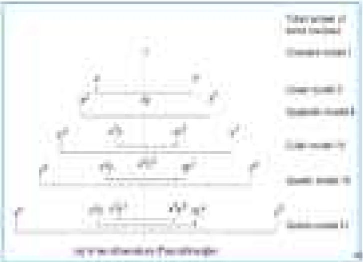
Three-dimensional case:

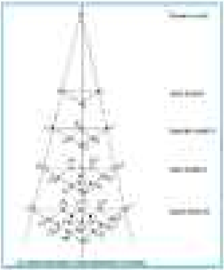
$$\phi(x,y,z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz$$

SELECTION OF THE ORDER OF THE INTERPOLATION POLYNOMIAL

[illegible]

1. [illegible]
2. [illegible]
3. [illegible]





$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

$$\{u\} = [A]\{a\}$$

$$\{a\} = [A]^{-1}\{u\}$$

$$\{u\} = [\phi][A]^{-1}\{u\}$$

$$\{u\} = [N]\{u\}$$

$$[N] = [\phi][A]^{-1}$$

$$[A]^{-1} = \frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}$$

$$[N] = [\phi][A]^{-1}$$

$$= \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{l} & \frac{1}{l} \end{bmatrix}$$

$[N] = \left[1 - \frac{x}{l} \quad \frac{x}{l} \right]$ → shape function

$N_1 = 1 - \frac{x}{l}$ and $N_2 = \frac{x}{l}$

Quadratic line element

$u = a_0 + a_1 x + a_2 x^2$ —— (1)

[illegible]

$\{u\} = [\;1 \quad x \quad x^2\;] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix}$

$\{u\} = [X]\{a\}$ —— (2)

[illegible]

[illegible]

Development of stiffness matrix

$$\varepsilon = \frac{\partial u}{\partial x} \qquad \boxed{\varepsilon = [B]\{q\}}$$

$$\left[-\frac{3}{l} + \frac{4x}{l^2} \quad \frac{4}{l} - \frac{8x}{l^2} \quad -\frac{1}{l} + \frac{4x}{l^2} \right] \{q\}$$

$$\sigma = [D][B]\{q\}$$

$$[K] = \int_V [B]^T [D][B] \, dV$$

$$[K] = \int_0^l \begin{bmatrix} -\frac{3}{l} + \frac{4x}{l^2} \\ \frac{4}{l} - \frac{8x}{l^2} \\ -\frac{1}{l} + \frac{4x}{l^2} \end{bmatrix} [E] \left[-\frac{3}{l} + \frac{4x}{l^2} \quad \frac{4}{l} - \frac{8x}{l^2} \quad -\frac{1}{l} + \frac{4x}{l^2} \right] A \, dx$$

$$[K] = \frac{EA}{3l} \int_0^l [\ \ ] \, dx$$

[illegible]

[illegible]

[illegible]

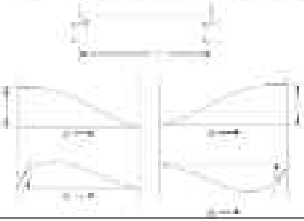
[illegible]

Final stiffness matrix

$$[K] = \frac{AE}{L}\begin{bmatrix} \frac{7}{3} & \frac{1}{3} & -\frac{8}{3} \\ \frac{1}{3} & \frac{7}{3} & -\frac{8}{3} \\ -\frac{8}{3} & -\frac{8}{3} & \frac{16}{3} \end{bmatrix}$$

Using polynomial functions (generalized coordinates)

Hermite shape functions for a two noded beam element



In this case C^1 – continuity is to be satisfied

where

$$\{d\}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \qquad \theta_1 = \frac{\partial w_1}{\partial x} \qquad \theta_2 = \frac{\partial w_2}{\partial x}$$

[illegible]

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

$$\theta = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

[illegible]

[illegible]

$$x_1 = 0$$
$$x_2 = l$$
$$\therefore w_1 = \alpha_1$$
$$\theta_1 = \alpha_2$$

$$w_2 = \alpha_1 + \alpha_2 l + \alpha_3 l^2 + \alpha_4 l^3$$
$$\theta_2 = \alpha_2 + 2\alpha_3 l + 3\alpha_4 l^2$$

$$\{d\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

[illegible]

$$\therefore \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore v = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l^2} & \frac{-2}{l} & \frac{3}{l^2} & \frac{-1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} & x - \frac{2x^2}{l} + \frac{x^3}{l^2} & \frac{3x^2}{l^2} - \frac{2x^3}{l^3} & -\frac{x^2}{l} + \frac{x^3}{l^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \{d\}_e = [N]\{d\}_e$$

where:

$$[N] = [N_1 \; N_2 \; N_3 \; N_4]$$

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \qquad N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \qquad N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

[illegible]

[illegible]

Natural Coordinates in Two Dimensions

$$L_1 + L_2 + L_3 = 1$$
$$L_1 x_1 + L_2 x_2 + L_3 x_3 = x$$
$$L_1 y_1 + L_2 y_2 + L_3 y_3 = y$$

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

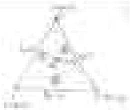
$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \frac{2A_1}{2A} \\ \frac{2A_2}{2A} \\ \frac{2A_3}{2A} \end{Bmatrix} = \frac{1}{2A} \{[illegible]\}$$

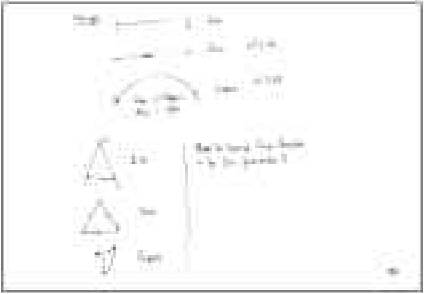
The Closed form integration for the ...

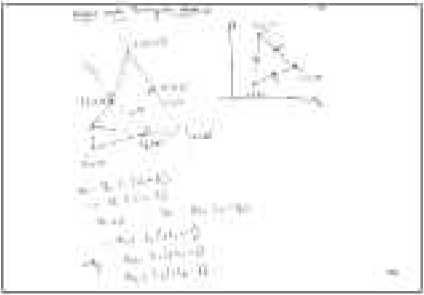
$$\int_A L_1^p\, L_2^q\, L_3^r\, dA = \frac{p!\, q!\, r!}{(p+q+r+2)!}\, 2A$$

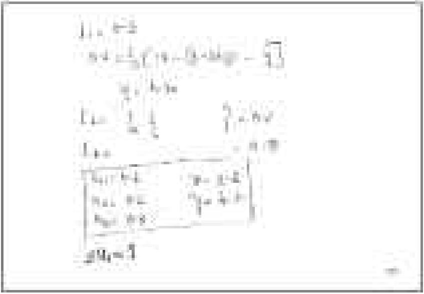
Solution: (a) $\int_A L_1\, L_2\, L_3\, dA$

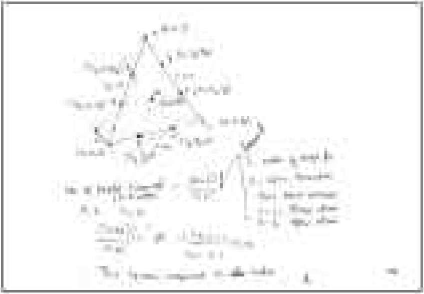
(b) $\int_A L_1^2\, L_2\, dA$

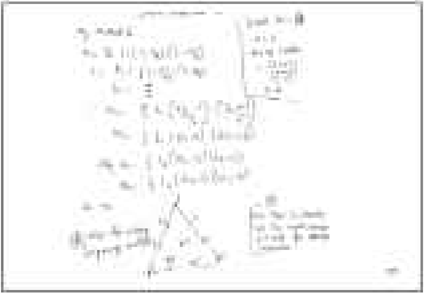


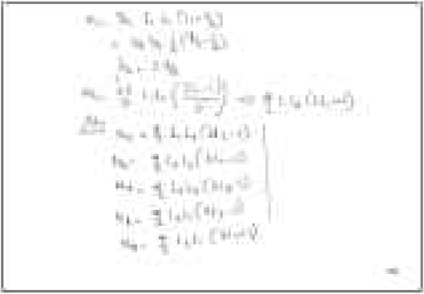


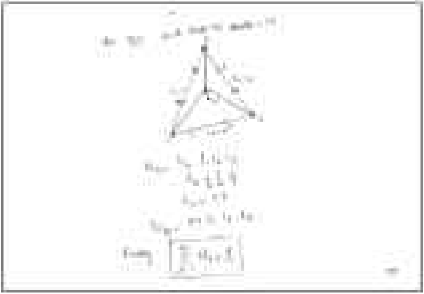


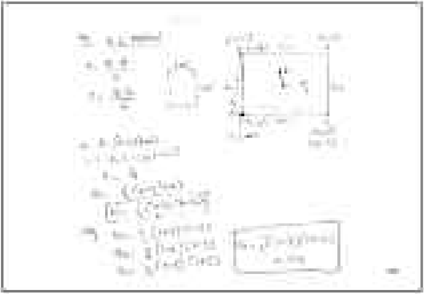


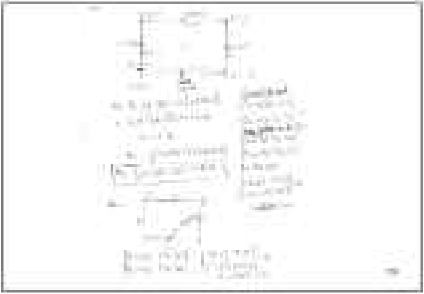


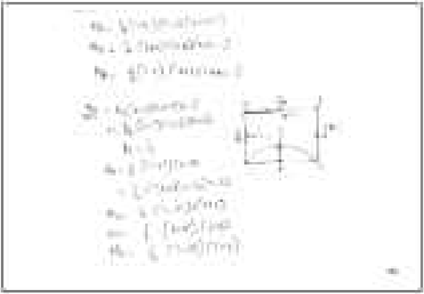


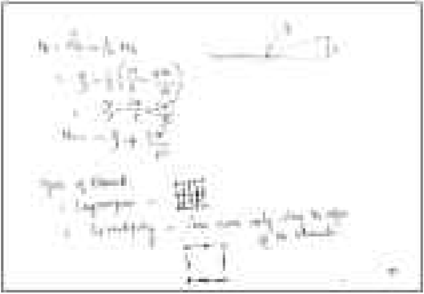


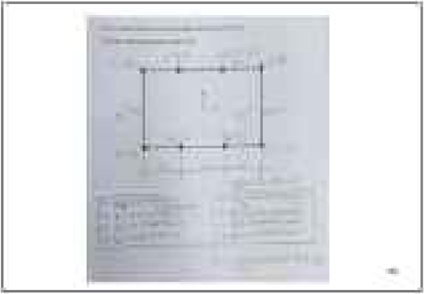












Convergence Requirements

1. The displacement function must be continuous within the element. This condition can easily be satisfied by choosing polynomials for the displacement model.

[illegible]

2. The displacement function must be capable of representing rigid body displacements of the element. That is when the nodes are given such displacements corresponding to a rigid body motion, the element should not experience any strain and hence leads to zero nodal forces. The constant terms in the polynomials used for the displacement models would usually ensure this condition.

3. The displacement function must be capable of representing constant strain states within the element. The reason for this requirement can be understood if we imagine the condition when the body or structure is divided into smaller and smaller elements. As these elements approach infinitesimal size, the strains in each element also approach constant values. Hence the assumed displacement function should include terms for representing constant strain states. For one, two and three-dimensional elasticity problems, the linear terms present in the polynomial satisfy the requirement. However, in the case of beam, plate and shell elements, this condition will be referred to as 'constant curvature' instead of 'constant strains'. The difficulty of satisfying this condition in these situations would be discussed later.

NODAL UNKNOWNS

Variational Method
–Energy Method

$d = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$u = u(x)$

Relate u (x) to nodal displacements $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$.

Let u(x) vary linearly within the element

$u(x) = a + bx$

At $x = 0$ $\quad u(0) = u_1 = a$

At $x = L$ $\quad u(L) = u_2 = a + bL = u_1 + bL$

$\therefore b = \dfrac{u_2 - u_1}{L}$

$u(x) = u_1 + \dfrac{u_2 - u_1}{L}x = \left(1 - \dfrac{x}{L}\right)u_1 + \left(\dfrac{x}{L}\right)u_2$

$\left(1 - \dfrac{x}{L}\right) = N_1(x)$ $\qquad$ $\left(\dfrac{x}{L}\right) = N_2(x)$

$\Rightarrow u(x) = N_1(x)\,u_1 + N_2(x)\,u_2$

$= \sum_{i=1}^{2} N_i(x)\,u_i$

where N_i = linear shape functions

or $u(x) = [N_1(x) \quad N_2(x)]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$\mathbf{N} = [N_1(x) \quad N_2(x)]$ $\qquad$ Shape Function Matrix

$u(x) = \mathbf{N}\mathbf{d}$ $\qquad$ → Discretisation

If $\hat{u}(x)$ approximates $u(x)$,

Error $= \hat{u}(x) - u(x)$

Strains

$$
\begin{aligned}
\varepsilon_x &= \frac{du(x)}{dx} \\
&= \frac{d}{dx}u(x) \\
&= \frac{d}{dx}\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\
&= \mathbf{LNd} \\
&= \mathbf{B}(x)\mathbf{d} \quad [\mathbf{B}(x) \rightarrow \text{Strain Matrix}] \\
&= \begin{bmatrix} \frac{dN_1(x)}{dx} & \frac{dN_2(x)}{dx} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\
\varepsilon &= \mathbf{Bd}
\end{aligned}
$$

Constitutive Relation

$\sigma_x = E\varepsilon_x$

In general $\sigma = \mathbf{E}\varepsilon$

Minimum Potential Energy

The Total Potential Energy of a system is:

$\Pi = U - W$

$\delta\Pi = \delta(U) - \delta(W) = 0$ (Stationary)

where U is strain energy stored in the body during deformation and

W is the work done by the external loads

We have established

$u = \mathbf{Nd}$ $\quad \varepsilon = \mathbf{Bd}$ $\quad \sigma = \mathbf{E}\varepsilon$

In the above, we have discretized displacement and strain expressions

Variational Statement

Linear element

$$\delta\Pi = \int_0^L (\delta\varepsilon)^T \sigma \, dx - \int_0^L (\delta u)^T p \, dx = 0$$

$$= \int_0^L \left[(B\,\delta u)^T E\,(B u) \right] dx - \int_0^L (N\,\delta u)^T p \, dx$$

$$= (\delta u)^T \left(\int_0^L B^T E B \, dx \right) u - (\delta u)^T \left(\int_0^L N^T p \, dx \right)$$

$$= (\delta u)^T \left[\left(\int_0^L B^T E B \, dx \right) u - \left(\int_0^L N^T p \, dx \right) \right]$$

$$\therefore \delta\Pi = 0 \Rightarrow \left(\int_0^L B^T E B \, dx \right) u - \left(\int_0^L N^T p \, dx \right) = 0$$

$$\therefore \; k u = P$$

$$k = \int_0^L B^T E B A \, dx \; ; \; P = \int_0^L N^T p \, dx$$

$$N_1 = \left(1 - \frac{x}{L}\right), \; N_2 = \frac{x}{L}$$

$$\frac{dN_1}{dx} = -\frac{1}{L}, \; \frac{dN_2}{dx} = \frac{1}{L}, \; dv = A\,dx$$

$$k = \int_0^L \left(\begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} AE \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \right) dx = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

Uniform Load

$$P = \int_0^L \begin{pmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{pmatrix} q \, dx$$

$$= \begin{pmatrix} \frac{qL}{2} \\ \frac{qL}{2} \end{pmatrix}$$

Bar Element

L	Length
A	Cross-sectional area
E	Elastic modulus
$u = u(x)$	Displacement
$\sigma = \sigma(x)$	Stress

Strain-displacement relation:

$$\varepsilon = \frac{du}{dx} \quad (1)$$

Stress-strain relation:

$$\sigma = E\varepsilon \quad (2)$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

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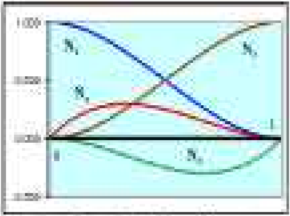
Example

Beam Element

Simple Beam Element



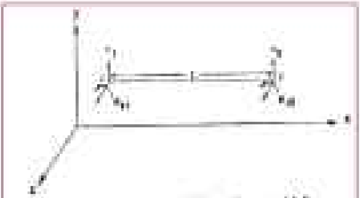
Shape Functions for Beam Element

[illegible]

[illegible]

Two-Dimensional Beam Element

Two-dimensional beam element with six degrees of freedom



Two-dimensional beam element with four degrees of freedom

Development of Shape Function For a Bending Element

[illegible]

$$w = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2$$

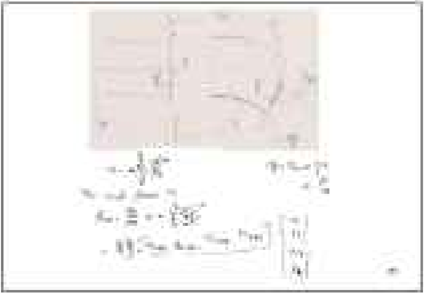
$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$$

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}$$

$$N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

[illegible]

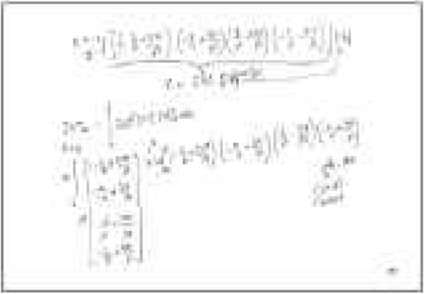


$$\frac{dN_1}{dx} = -\frac{6x}{l^2} + \frac{6x^2}{l^3} \; ; \; \frac{d^2N_1}{dx^2} = -\frac{6}{l^2} + \frac{12x}{l^3}$$

$$\frac{dN_2}{dx} = 1 - \frac{4x}{l} + \frac{3x^2}{l^2} \; ; \; \frac{d^2N_2}{dx^2} = -\frac{4}{l} + \frac{6x}{l^2}$$

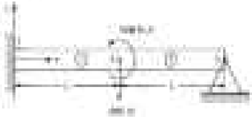
$$\frac{dN_3}{dx} = \frac{6x}{l^2} - \frac{6x^2}{l^3} \; ; \; \frac{d^2N_3}{dx^2} = \frac{6}{l^2} - \frac{12x}{l^3}$$

$$\frac{dN_4}{dx} = -\frac{2x}{l} + \frac{3x^2}{l^2} \; ; \; \frac{d^2N_4}{dx^2} = -\frac{2}{l} + \frac{6x}{l^2}$$



[illegible]

Beam Analysis : 1



[illegible]

stiffness matrices

$$k^{(1)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$k^{(2)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The total stiffness matrix

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

boundary conditions:

$$\phi_1 = 0 \qquad v_1 = 0 \qquad v_3 = 0$$

$$\begin{Bmatrix} -1000 \\ 1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

where $F_{2y} = -1000$ N, $M_2 = 1000$ N·m, and $M_3 = 0$

determine (1) the slopes at 2 and 3 and vertical deflection at the midpoint of the distributed load

Beam Analysis : 2

$$\frac{EI}{l^3} = \frac{(200 \times 10^9)(4 \times 10^{-6})}{1^3} = 8 \times 10^5 \text{ N/m}$$

$$\mathbf{k}^1 = \mathbf{k}^2 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$8 \times 10^5 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ +1000 \end{Bmatrix}$$

[illegible]

For element 2, $q_1 = 0$, $q_2 = Q_4$, $q_3 = 0$, and $q_4 = Q_6$.

To get vertical deflection at the mid point of the element, use

[illegible]

$$H_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$$

$$H_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$H_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}$$

$$H_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

$H_1 = 0.5$, $H_2 = 0.125$

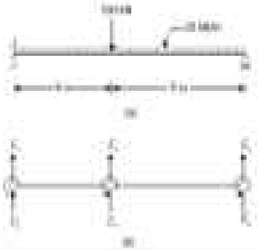
$H_3 = 0.5$, $H_4 = -0.125$

$v = H_2 q_2 + H_4 q_4$

$$v = -0.893 \times 10^{-4} \text{ m}$$

$$= -0.0893 \text{ mm}$$

[illegible]



$$[F] = \begin{Bmatrix} [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \end{Bmatrix}$$ [illegible]

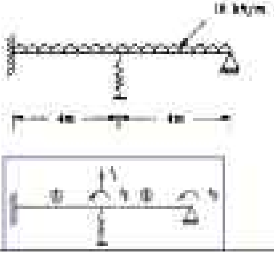
Answer

$v_2 = -0.0322$ m

$T_2 = -0.003$ rad

$T_3 = 0.006$ rad

Beam Analysis : 3



$$[k_1] = \begin{bmatrix} 75 & 150 & -75 & 150 \\ & 400 & -150 & 200 \\ \text{sym.} & & 75 & -150 \\ & & & 400 \end{bmatrix}$$

$$[k_2] = \begin{bmatrix} 75 & 150 & -75 & 150 \\ & 400 & -150 & 200 \\ \text{sym.} & & 75 & -150 \\ & & & 400 \end{bmatrix}$$

$$\{Q_1\} = -[S_{21}] \times \begin{Bmatrix} -\frac{w l}{2} \\ -\frac{w l^2}{12} \\ -\frac{w l}{2} \\ \frac{w l^2}{12} \end{Bmatrix} = \begin{matrix} 1 & 1 \\ 2 & 2 \\ 1 & 3 \\ 3 & 4 \end{matrix} \begin{Bmatrix} -20 \\ -20 \\ -20 \\ 20 \end{Bmatrix}$$

$$\{Q_2\} = -[S_{22}] = \begin{matrix} 1 & 1 \\ 2 & 2 \\ 4 & 3 \\ 3 & 4 \end{matrix} \begin{Bmatrix} -60 \\ -20 \\ -50 \\ 20 \end{Bmatrix}$$

[illegible]

$$\begin{bmatrix} 500 & 0 & -100 \\ & 800 & 200 \\ \text{sym} & & 400 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} -60 \\ 0 \\ 20 \end{Bmatrix}$$

Solving the above equations we get,

[illegible]

[illegible]

[illegible]

The final end moments for each member are calculated using

$$\{F\} = [k]\{\delta\} + \{F_f\}$$

Member 1

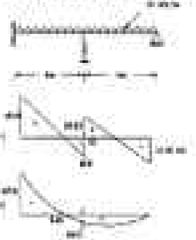
[illegible]

Member 2

[illegible]

[illegible]

Using the above values of final end moments for the members 1 and 2, the shear force and bending moment diagrams are drawn as shown



[illegible]

Beam Analysis : 4

Similarly

$$k_2 = 8\begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}\begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

(column dof: 3, 4, 5, 6)

$$\therefore [K] = 8\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

(column dof: 1, 2, 3, 4, 5, 6)

Consistent load vector is given by

$$F = \begin{Bmatrix} \frac{wL}{2} \\ \frac{wL^2}{12} \\ \frac{wL}{2} \\ -\frac{wL^2}{12} \end{Bmatrix} \quad \therefore F^{(1)} = \begin{Bmatrix} \frac{(-24)\times 5}{2} \\ \frac{(-24)\times 5^2}{12} \\ \frac{(-24)\times 5}{2} \\ \frac{-(-24)\times 5^2}{12} \end{Bmatrix} = \begin{Bmatrix} -60 \\ -50 \\ -60 \\ 50 \end{Bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\{f^{(2)}\} = \begin{Bmatrix} \frac{(-24)\times 5}{2} \\ \frac{(-24)\times 5^2}{12} \\ \frac{(-24)\times 5}{2} \\ \frac{-(-24)\times 5^2}{12} \end{Bmatrix} = \begin{Bmatrix} -60 \\ -50 \\ -60 \\ 50 \end{Bmatrix} \qquad \{F\} = \begin{Bmatrix} -60 \\ -25 \\ -60 \\ 25 \\ -60 \\ 50 \end{Bmatrix}$$

- The stiffness equation is:

$$E\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \begin{Bmatrix} -60 \\ -25 \\ -120 \\ -25 \\ -60 \\ 50 \end{Bmatrix}$$

Boundary conditions:

In the above problem the boundary conditions are

$$d_1 = d_2 = d_5 = d_6 = 0$$

$$k\begin{bmatrix}200 & 50\\ 50 & 100\end{bmatrix}\begin{Bmatrix}\delta_4\\ \theta_4\end{Bmatrix}=\begin{Bmatrix}-25\\ 50\end{Bmatrix}$$

$$400\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\delta_4\\ \theta_4\end{Bmatrix}=\begin{Bmatrix}-25\\ 50\end{Bmatrix}$$

$$\begin{Bmatrix}\delta_4\\ \theta_4\end{Bmatrix}=\frac{1}{400}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}^{-1}\begin{Bmatrix}-25\\ 50\end{Bmatrix}$$

$$=\frac{1}{400}\times\frac{1}{8-1}\begin{bmatrix}2 & -1\\ -1 & 4\end{bmatrix}\begin{Bmatrix}-25\\ 50\end{Bmatrix}=\frac{1}{2800}\begin{Bmatrix}-100\\ 225\end{Bmatrix}\quad \text{Answer}$$

End reactions:

$$\begin{Bmatrix}R_3\\ M_3\\ R_4\\ M_4\end{Bmatrix}=8\begin{bmatrix}12 & 30 & -12 & 30\\ 30 & 100 & -30 & 50\\ -12 & -30 & 12 & -30\\ 30 & 50 & -30 & 100\end{bmatrix}\begin{Bmatrix}0\\ 0\\ -100\\ 225\end{Bmatrix}\frac{1}{2800}-\begin{Bmatrix}-25\\ -21\\ -25\\ 21\end{Bmatrix}$$

$$R_1 = R_2 = R_3 = 0 \text{ and } R_4 = \frac{100}{2800}\begin{Bmatrix}33.83\\ 19.14\\ -9.175\\ -4.175\end{Bmatrix}\quad \text{Answer}$$

For element 2

[illegible]

answer

Deflection at mid span

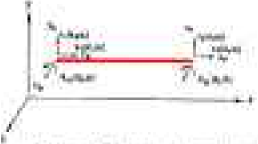
$$v = [N_1 \quad N_2 \quad N_3 \quad N_4]\{d\}_e$$

For element 1, [illegible] m = [illegible] mm

For element 2, = 27.9 mm downward answer

[illegible]

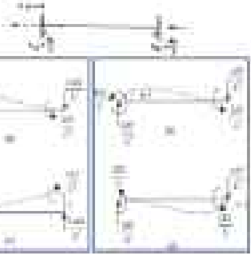
Stiffness Matrix for a Two Dimensional Beam (Element with Six Degrees of Freedom)



Two dimensional beam element with six degrees of freedom

[illegible]

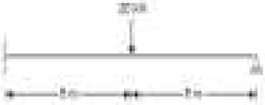
$$
[k]=\begin{bmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0\\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2}\\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L}\\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0\\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2}\\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
$$

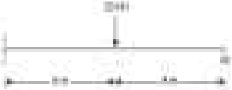


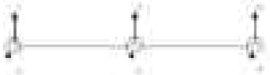
Determine: 1. Deflection under load
2. Shear force and bending moment of each span
3. Reactions at supports

$E = 200\ GPa = 200 \times 10^{9}\ N/m^2$

moment of inertia = $24 \times 10^{-6}\ m^4$







Nodal displacement vector is

$$\{u\}^T = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6]$$

Stiffness matrix for an element

$$k_1 = \frac{EI}{L^3}\begin{bmatrix} [illegible] \end{bmatrix}$$

$$k_1 = \frac{6000}{125}\begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}$$

Similarly,

$$k_2 = \frac{6000}{125}\begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}$$

Global Stiffness Matrix

$$[K] = \frac{6000}{125}\begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$\frac{EI}{125}\begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix}\begin{Bmatrix} \Delta_{[illegible]} \\ \theta_4 \\ \theta_5 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

[illegible]

$$\begin{Bmatrix} \Delta_{[illegible]} \\ \theta_4 \\ \theta_5 \end{Bmatrix} = \frac{125}{EI}\frac{1}{248000}\begin{bmatrix} 17500 & 1500 & -6000 \\ 1500 & 1500 & -1200 \\ -6000 & -1200 & 4800 \end{bmatrix}\begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \frac{1}{EI}\begin{Bmatrix} -14.292 \\ -15.625 \\ 62.5 \end{Bmatrix} = \frac{1}{4000}\begin{Bmatrix} -14.292 \\ -15.625 \\ 62.5 \end{Bmatrix} = \begin{Bmatrix} [illegible] \\ [illegible] \times 10^{[illegible]} \\ 4.8832 \end{Bmatrix}$$

[illegible]

Prepared by
[illegible]

Shear Force and Bending Moment at midspan

Considering element (1), for mid span of beam

$$M = \frac{EI}{L^2}[\text{[illegible]}]\frac{1}{EI}\begin{bmatrix} 0 \\ 0 \\ -442.[\text{illegible}] \\ -43.[\text{illegible}] \end{bmatrix}$$

$$= \frac{1}{[\text{illegible}]}[\text{[illegible]}]\begin{bmatrix} 0 \\ 0 \\ -[\text{illegible}] \\ -[\text{illegible}] \end{bmatrix} = 11.[\text{illegible}]\ \text{KN.m}$$

[illegible]

$$V = \frac{EI}{L^3}[\text{[illegible]}]\frac{1}{EI}\begin{bmatrix} 0 \\ 0 \\ -[\text{illegible}] \\ -[\text{illegible}] \end{bmatrix} = [\text{illegible}]\ \text{kN}$$

[illegible]

End Reactions

At support on left hand side, from element 1.

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & [\text{illegible}] & -12 & [\text{illegible}] \\ [\text{illegible}] & [\text{illegible}] & [\text{illegible}] & [\text{illegible}] \end{bmatrix}\frac{1}{EI}\begin{bmatrix} 0 \\ 0 \\ -[\text{illegible}] \\ -[\text{illegible}] \end{bmatrix} - \begin{bmatrix} [\text{illegible}] \\ [\text{illegible}] \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} [\text{illegible}] \\ [\text{illegible}] \end{bmatrix}$$

[illegible]

Considering element 2

[illegible]

[illegible]

Beam Example

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

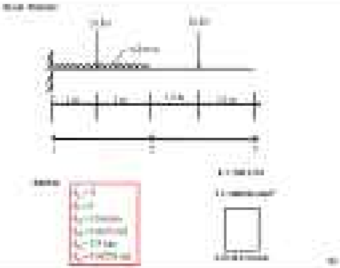
$$[illegible]$$

[illegible]

$$[illegible]$$

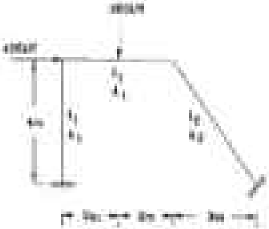
[illegible]

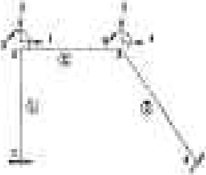
$$[illegible]$$

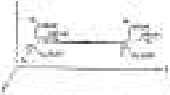


2D Frame Analysis

Plane Frame Analysis







Two-dimensional beam element with six degrees of freedom

[illegible]

Member 1

[illegible]

[illegible]

[illegible]

$$[T] = \begin{bmatrix} C_x & C_y & 0 & 0 & 0 & 0 \\ -C_y & C_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & 0 \\ 0 & 0 & 0 & -C_y & C_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_x = 0, \; C_y = 1.0$

$$[T] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The stiffness matrix [k] in global axes given by,

$$[k]_1 = [T]^T [k_e][T]$$

$$[k] = \begin{bmatrix} [illegible] & 0 & -[illegible] & -[illegible] & 0 & -[illegible] \\ 0 & -11 \times 10^6 & 0 & 0 & -11 \times 10^6 & 0 \\ -[illegible] & 0 & [illegible] & [illegible] & 0 & [illegible] \\ -[illegible] & 0 & [illegible] & [illegible] & 0 & [illegible] \\ 0 & -11 \times 10^6 & 0 & 0 & 11 \times 10^6 & 0 \\ -[illegible] & 0 & [illegible] & [illegible] & 0 & [illegible] \end{bmatrix}$$

Member 2

Since this member is oriented in the global directions, no transformation is required. The member stiffness matrix in the local directions is the same as in the case of member 1.

$$[k] = \begin{bmatrix} 75\times10^3 & 0 & 0 & -75\times10^3 & 0 & 0 \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \\ -75\times10^3 & 0 & 0 & 75\times10^3 & 0 & 0 \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \end{bmatrix}$$

The element load vector also does not require any transformation.

$$\{Q_f\} = \begin{Bmatrix} 0 \\ -\frac{wL}{2} \\ -\frac{wL^2}{12} \\ 0 \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -180 \\ -180 \\ 0 \\ -180 \\ 180 \end{Bmatrix}$$

Member 3

$[k_m] =$

[illegible]	0	0	[illegible]	0	0
0	[illegible]	[illegible]	0	[illegible]	[illegible]
0	[illegible]	[illegible]	0	[illegible]	[illegible]
[illegible]	0	0	[illegible]	0	0
0	[illegible]	[illegible]	0	[illegible]	[illegible]
0	[illegible]	[illegible]	0	[illegible]	[illegible]

$C_x = 0.8, \; C_y = -0.6$

$$[T] = \begin{bmatrix} 0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k_g] = [T]^T [k_m] [T]$$

$[k_g] =$

		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

$$\begin{bmatrix} [illegible] \end{bmatrix} \begin{Bmatrix} [illegible] \end{Bmatrix} = \begin{Bmatrix} [illegible] \end{Bmatrix}$$

$$\{r\} = \begin{Bmatrix} 0.2558 \\ [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \\ [illegible] \end{Bmatrix}$$

Member End Forces

$\{N\} = [k_m][d_m] + \{F_e\}$

where $[d_m] = [T][d]$

Member 1

$$[d_m] = \begin{bmatrix} [illegible] \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ [illegible] \\ [illegible] \times 10^{-4} \\ [illegible] \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ [illegible] \times 10^{-4} \\ [illegible] \\ [illegible] \end{Bmatrix}$$

$$\{f\} = [k_e]\{u_e\} + \{f_e\}$$

$$\{f\} = \begin{bmatrix} 15\times10^{5} & 0 & 0 & -15\times10^{5} & 0 & 0 \\ 0 & 800 & 900 & 0 & -800 & 900 \\ 0 & 900 & 2400 & 0 & -900 & 1200 \\ -15\times10^{5} & 0 & 0 & 15\times10^{5} & 0 & 0 \\ 0 & -800 & -900 & 0 & 800 & -900 \\ 0 & 900 & 1200 & 0 & -900 & 2400 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1.0072\times10^{-4} \\ -0.2575 \\ -0.0000 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -14.11 \\ 17.57 \\ 121.01 \\ 13.11 \\ -23.07 \\ 163.84 \end{Bmatrix}$$

[illegible]

Element 2

[illegible] $[k_e] = [k]$

$$\{f\} = \begin{bmatrix} 15\times10^{5} & 0 & 0 & -15\times10^{5} & 0 & 0 \\ 0 & 800 & 900 & 0 & -800 & 900 \\ 0 & 900 & 2400 & 0 & -900 & 1200 \\ -15\times10^{5} & 0 & 0 & 15\times10^{5} & 0 & 0 \\ 0 & -800 & -900 & 0 & 800 & -900 \\ 0 & 900 & 1200 & 0 & -900 & 2400 \end{bmatrix} \times \begin{Bmatrix} 0.3275 \\ 1.0072\times10^{-4} \\ -0.2575 \\ 0.2000 \\ 0.2128 \\ -0.7000 \end{Bmatrix} + \begin{Bmatrix} 5.8 \\ 2000 \\ 2083 \\ 5.8 \\ 2000 \\ -2083 \end{Bmatrix} = \begin{Bmatrix} 8586 \\ -31.1 \\ -1564.98 \\ -8469 \\ 213.1 \\ -1598.4 \end{Bmatrix}$$

[illegible]

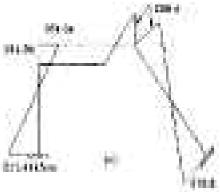
Member 2

$$\{d_2'\} = [T]\{d\}$$

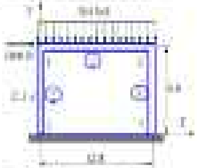
$$\{d_2'\} = \begin{bmatrix} 0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} [illegible] \\ [illegible] \\ [illegible] \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} [illegible] \times 10^{-4} \\ [illegible] \\ [illegible] \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

$$\{f'\} = [k_2'][d_2'] + [f_0']$$

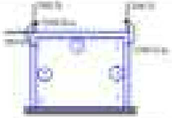
$$\{f'\} = \begin{bmatrix} [illegible] \times 10^{5} & 0 & 0 & [illegible] \times 10^{5} & 0 & 0 \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \\ [illegible] \times 10^{5} & 0 & 0 & [illegible] \times 10^{5} & 0 & 0 \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \\ 0 & [illegible] & [illegible] & 0 & [illegible] & [illegible] \end{bmatrix} \times \begin{Bmatrix} [illegible] \times 10^{-4} \\ [illegible] \\ [illegible] \\ 0.0 \\ 0.0 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} [illegible] \\ -114.4 \\ [illegible] \\ [illegible] \\ -114.4 \\ [illegible] \end{Bmatrix}$$



[illegible]



[illegible]



[illegible]

$$K = \begin{bmatrix} [illegible] \end{bmatrix}$$

[illegible]

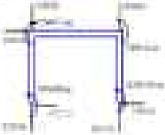
$$K = [illegible]$$

[illegible]

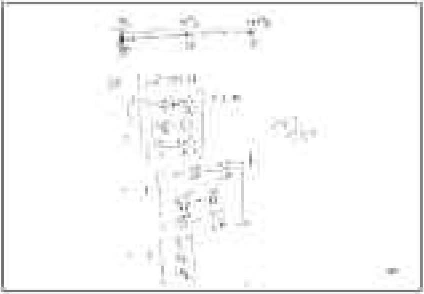
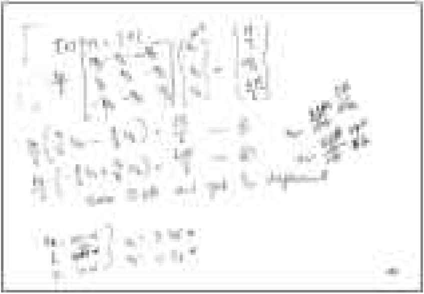
$$K = \begin{bmatrix} [illegible] \end{bmatrix}$$

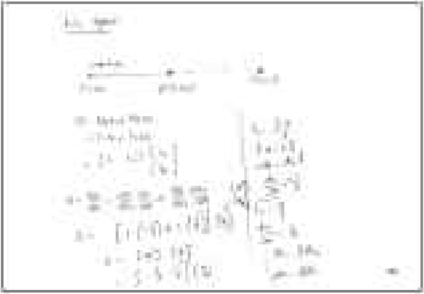
[illegible]

$$K = \begin{bmatrix} [illegible] \end{bmatrix}$$



Computation of Element Load Vectors

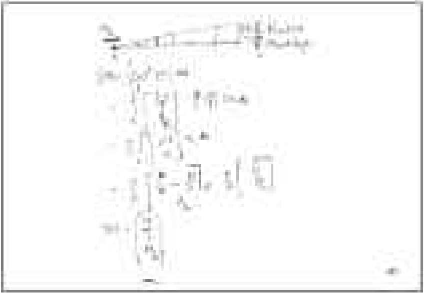


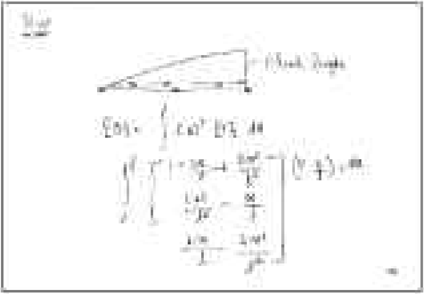


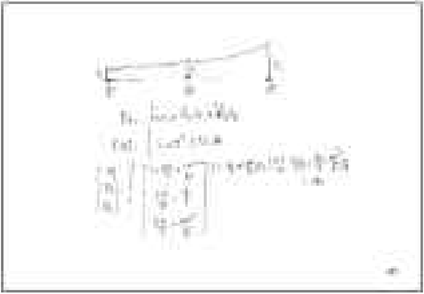
$[K] = \int [B]^T [D][B]\, dV$

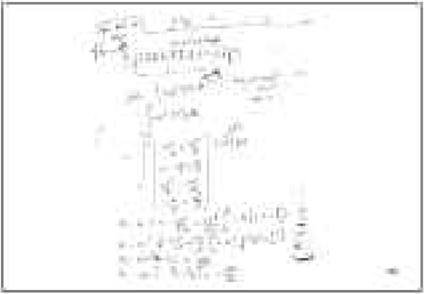
[illegible]

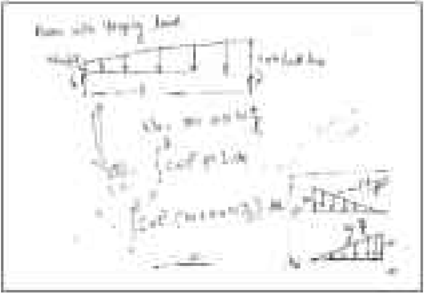
[illegible]

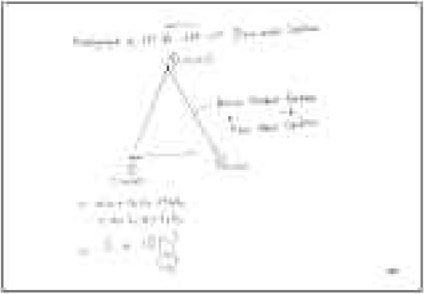


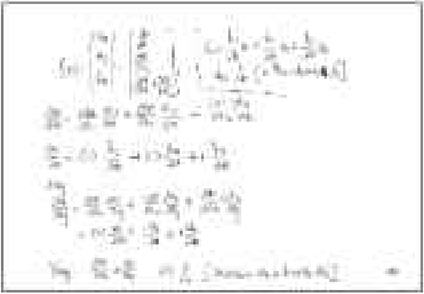


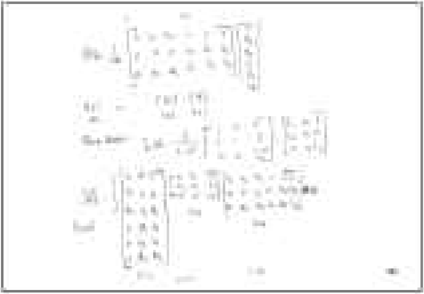


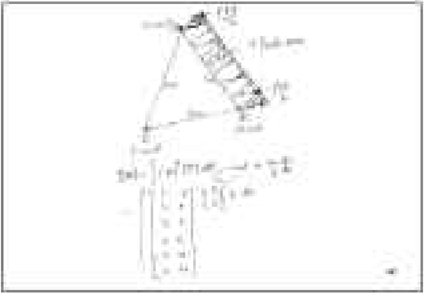


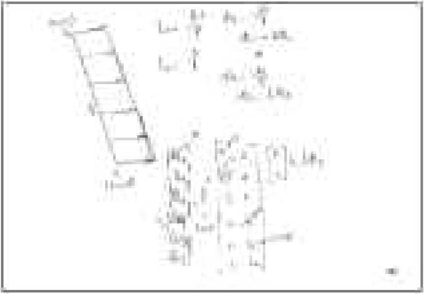


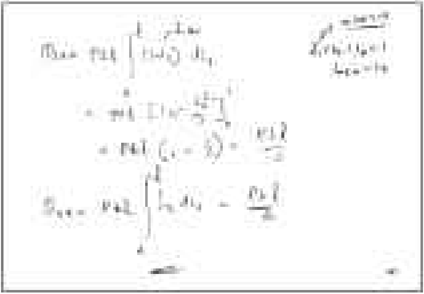


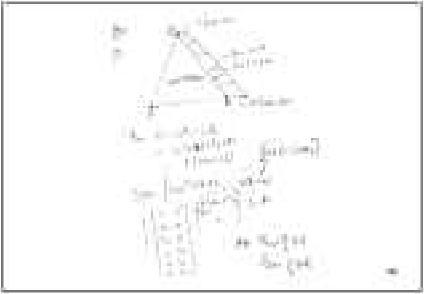












[illegible]

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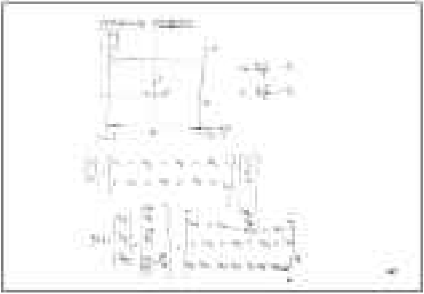

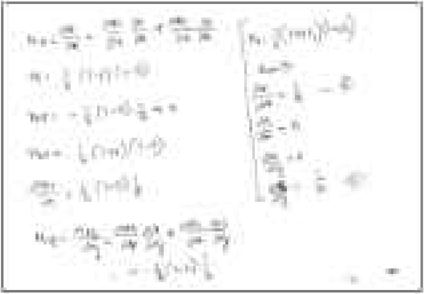

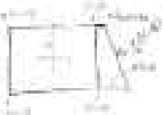

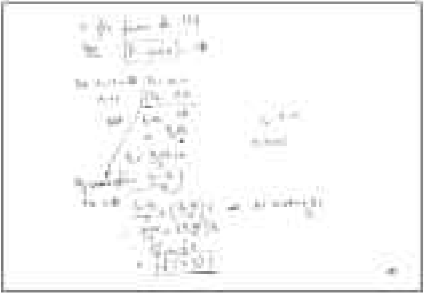

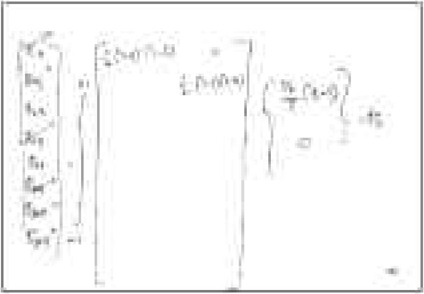

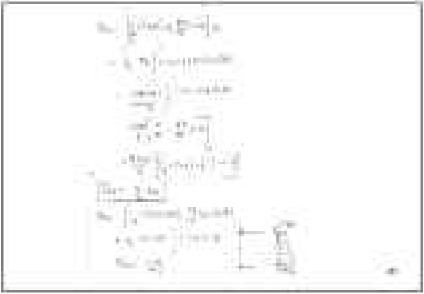

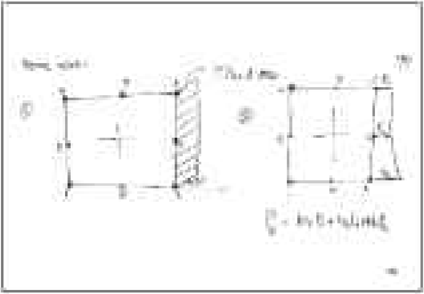

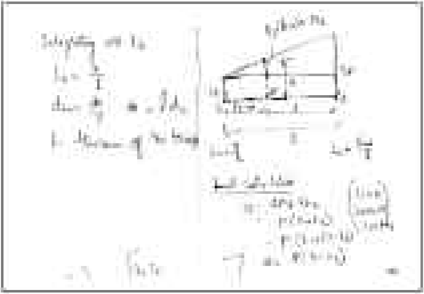

[illegible]

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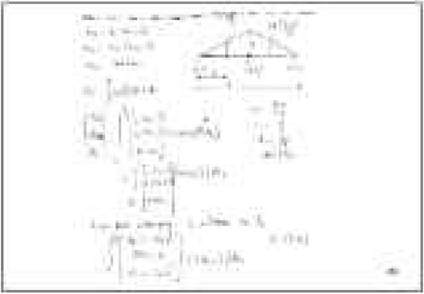
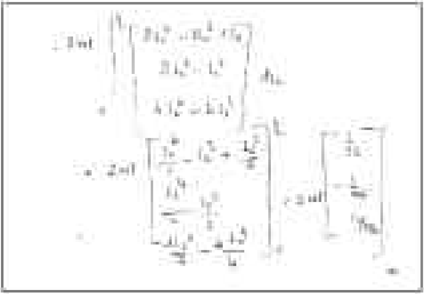
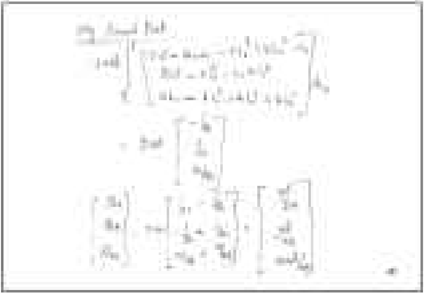
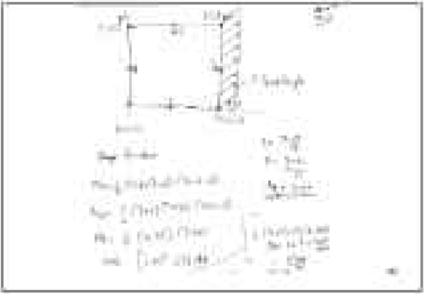
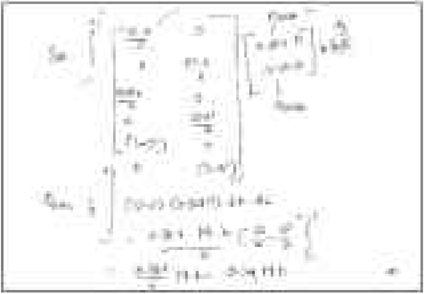
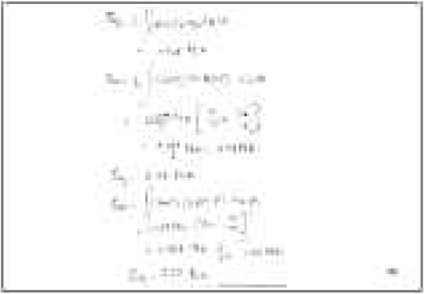












Isoparametric Element Formulation

Isoparametric Elements and Solution

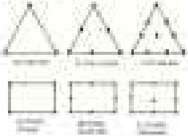
- Biggest breakthrough in the implementation of the finite element method is the development of an isoparametric element with capabilities to model structure (problem) geometries of any shape and size.
- The whole idea works on mapping.
  - The element in the real structure is mapped to an imaginary element in an ideal coordinate system
  - The solution to the stress analysis problem is easy and known for the 'imaginary' element
  - These solutions are mapped back to the element in the real structure.
  - All the loads and boundary conditions are also mapped from the real to the 'imaginary' element in this approach

Isoparametric Formulation

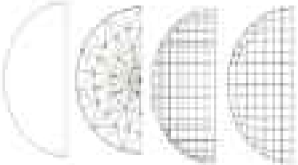
Same function that is used to define the element geometry is used to define the displacements within the element.

Necessity of Isoparametric Formulation

The family of 2-D elements discussed till now are of regular geometry (e.g. triangular and rectangular elements) having straight edges.

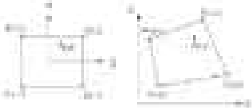


Analysis of any irregular geometry is difficult to use such element directly.



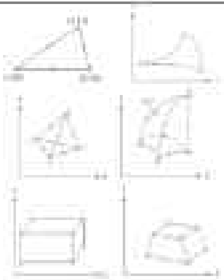
FE Discretization of the irregular

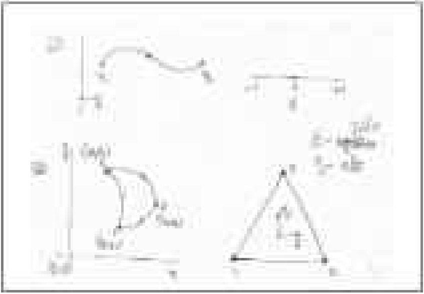
The Iso-parametric quadrilateral elements can be derived from rectangular elements using the concept of mapping.

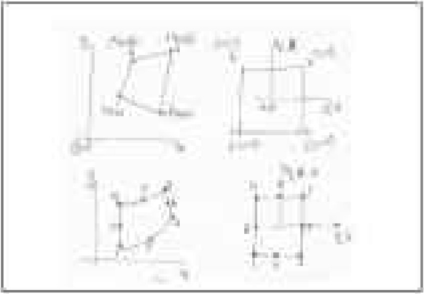


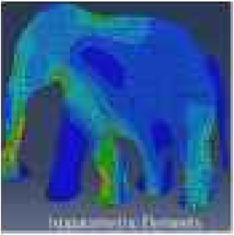
Using the concept of mapping regular triangular, rectangular or solid elements in natural coordinate system (known as parent element) can be transformed into global Cartesian coordinate system having arbitrary shapes (with curved edges or surfaces).

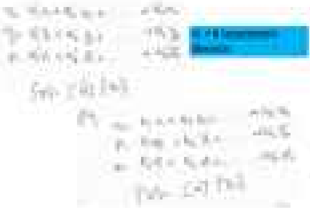
Isoparametric Element





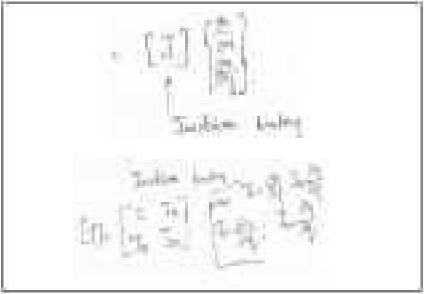






[illegible]

[illegible]



[illegible]

Inverse of Jacobian matrix

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

Differentiate N_i with ξ and η

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$

[illegible]

[illegible]

[illegible]

$$\frac{\partial u}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)u_1 + (1-\eta)u_2 + (1+\eta)u_3 - (1+\eta)u_4\right]$$

$$\frac{\partial u}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)u_1 - (1+\xi)u_2 + (1+\xi)u_3 + (1-\xi)u_4\right]$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)v_1 + (1-\eta)v_2 + (1+\eta)v_3 - (1+\eta)v_4\right]$$

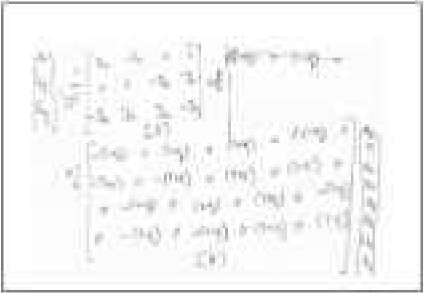
$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)v_1 - (1+\xi)v_2 + (1+\xi)v_3 + (1-\xi)v_4\right]$$

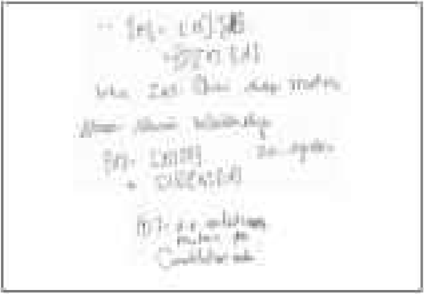
[illegible]

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \frac{1}{4} [\text{illegible}] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

[illegible]

Prepared by
[illegible]





For Plane – Stress Condition

$$[D] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\nu}{2}\right) \end{bmatrix}$$

Plane – Strain Condition

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$

where $\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$, $\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$

Determine the cartesian coordinate of point P which has ξ = 0.5 and η = 0.6 as shown in the figure.

$N_1 = \frac{1}{4}(1-\xi)(1-\eta) = \frac{1}{4}(1-0.4)(1-[illegible]) = [illegible]$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta) = \frac{1}{4}(1+[illegible])(1-[illegible]) = [illegible]$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta) = [illegible]$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta) = [illegible]$

Find x and y values:

$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$

$= [illegible]$

$= [illegible]$

$y = [illegible]$

$= [illegible]$

$= [illegible]$

The Cartesian Coord of point P [illegible] ([illegible])

Find cartesian coordinate of the point defined by (r, s) as (0.5, 0.5)

The cartesian coordinate of point P

For the isoparametric quadrilateral element shown in the figure, find the local coordinates of point P, where the cartesian coordinates are

[illegible]

[illegible]

Since the value of ξ and η ranges from -1 to 1, the I.C of points may be taken as (1, 0)

Find also the ξ, η values at the nodal locations (2, 1)

[illegible]

[illegible]

Evaluate the [J] and [B] at local coordinates [illegible] for the linear quadrilateral element with its global coordinates as shown in the figure.

Find the Jacobian at point (1/2, 1/2)

[illegible]

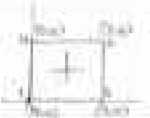
[illegible]

[illegible]

For a rectangular element shown in figure, determine the following at [illegible]

a) Jacobian Matrix
b) strain displacement matrix
c) [illegible]
d) [illegible]

Take E = [illegible]
Assume Plane stress condition



Assume the nodal displacements {q} = [illegible]

[illegible]

[illegible]

(3) To find element strain

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [B]\{u\}$$

(nodal displacement obtained)

$$\{\varepsilon\} = \frac{1}{[illegible]} \begin{bmatrix} [illegible] \end{bmatrix} \begin{Bmatrix} [illegible] \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} [illegible] \\ [illegible] \\ [illegible] \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$ [illegible]

(4) To find element Stress

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D][B]\{u\}$$

Here [D] = Constitutive matrix [Elasticity matrix] for plane stress condition

Solution

Coordinate Transformation

The geometry of an element may be expressed in terms of the interpolation functions as follows:

$$x = N_1 x_1 + N_2 x_2 + \dots + N_n x_n = \sum_{i=1}^{n} N_i x_i$$

$$y = N_1 y_1 + N_2 y_2 + \dots + N_n y_n = \sum_{i=1}^{n} N_i y_i$$

$$z = N_1 z_1 + N_2 z_2 + \dots + N_n z_n = \sum_{i=1}^{n} N_i z_i$$

Where
n = No. of Nodes
N_i = Interpolation functions
x_i, y_i, z_i = Coordinates of Nodal Points of the Element

[illegible]

One can also express the field variable variation in the element as

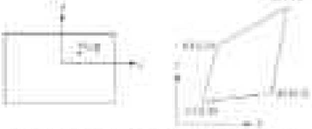
$$\phi(\xi,\eta,\zeta) = \sum_{i=1}^{n} N_i(\xi,\eta,\zeta)\,\phi_i$$

As the same shape functions are used for both the field variable and description of element geometry, the method is known as isoparametric mapping.

The element defined by such a method is known as an isoparametric element.

[illegible]

Determine the Cartesian coordinate of the point P (ξ= 0.8, η= 0.6) as shown in Fig.



As mentioned above, the relation between two coordinate systems can be represented through their interpolation functions. Therefore, the values of the interpolation function at point P will be:

[illegible]

[illegible]

Thus the coordinate of point P in Cartesian coordinate system can be calculated as

[illegible]

Thus the coordinate of point P [illegible] in Cartesian coordinate system will be [illegible]

[illegible]

Concept of Jacobian Matrix

- A variety of derivatives of the interpolation functions with respect to the global coordinates are necessary to formulate the element stiffness matrices.

- As the both element geometry and variation of the shape functions are represented in terms of the natural coordinates of the parent element, some additional mathematical obstacle arises.
- For example, in case of evaluation of the strain vector, the operator matrix is with respect to x and y, but the interpolation function is with ξ and η.

The relationship between the coordinate systems can be computed by using the chain rule of partial differentiation as

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta}$$

The above equations can be expressed in matrix form as well.

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

The matrix [J] is denoted as the Jacobian matrix as

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

As we know, $x = \sum_{i=1}^{n} N_i x_i$

where n is the number of nodes in an element

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial \left(\sum_{i=1}^{n} N_i x_i\right)}{\partial \xi} = \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i$$

Similarly we can calculate for the terms J_{12}, J_{21} and J_{22} [illegible]

Thus,

$$[J] = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

we can write

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

Considering $\begin{bmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{bmatrix}$ as the [illegible]

we have,

$$\frac{\partial}{\partial x} = J^*_{11} \frac{\partial}{\partial \xi} + J^*_{12} \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = J^*_{21} \frac{\partial}{\partial \xi} + J^*_{22} \frac{\partial}{\partial \eta}$$

Similarly, for three dimensional case,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix}$$

Where,

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \qquad [J] = \sum_{i=1}^{n} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} x_i & \frac{\partial N_i}{\partial \xi} y_i & \frac{\partial N_i}{\partial \xi} z_i \\ \frac{\partial N_i}{\partial \eta} x_i & \frac{\partial N_i}{\partial \eta} y_i & \frac{\partial N_i}{\partial \eta} z_i \\ \frac{\partial N_i}{\partial \zeta} x_i & \frac{\partial N_i}{\partial \zeta} y_i & \frac{\partial N_i}{\partial \zeta} z_i \end{bmatrix}$$

Evaluation of Stiffness Matrix of 2-D Isoparametric Elements

For two dimensional plane stress/strain formulation, the strain vector can be represented as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} J_{11}^{*} \frac{\partial u}{\partial \xi} + J_{12}^{*} \frac{\partial u}{\partial \eta} \\ J_{21}^{*} \frac{\partial v}{\partial \xi} + J_{22}^{*} \frac{\partial v}{\partial \eta} \\ J_{21}^{*} \frac{\partial u}{\partial \xi} + J_{22}^{*} \frac{\partial u}{\partial \eta} + J_{11}^{*} \frac{\partial v}{\partial \xi} + J_{12}^{*} \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

[illegible]

$$\{\varepsilon\} = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

[illegible]

$$u = \sum_{i=1}^{n} N_i u_i$$ [illegible]

Thus:

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_n \end{Bmatrix}$$

$$\{\varepsilon\} = [\text{illegible}]\,[\text{illegible}]\,\{\text{illegible}\}$$

Or,

$$\{\varepsilon\} = [B]\{d\}$$

Where {d} is the nodal displacement vector

[B] is known as [illegible] matrix and can be obtained.

$$[B] = [\text{illegible}]\,[\text{illegible}]$$

It is necessary to [illegible] from Cartesian to the natural coordinates as well for evaluation of the [illegible] isoparametric formulation.

Prepared by

The differential area relationship can be established from advanced calculus and the elemental area in Cartesian coordinates can be represented in terms of area in natural coordinates as:

$$dA = dx\,dy = |J|\,d\xi\,d\eta$$

Here $|J|$ is the determinant of the Jacobian matrix.

The stiffness matrix for a two dimensional element may be expressed as:

$$[k] = \iint [B]^T [D][B]\, t\, dx\, dy = t \iint [B]^T [D][B]\, dA$$

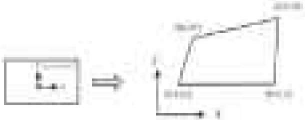
$[B]$ is the strain-displacement relationship matrix

t is the thickness of the element

The above expression in Cartesian coordinate system can be changed to the natural coordinate system as follows to obtain the elemental stiffness matrix

$$[k] = t\int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\,|J|\,d\xi\,d\eta$$

Calculate the Jacobian matrix and the strain displacement matrix for four noded two dimensional elements corresponding to the gauss point (0.57735, 0.57735)(quadrilateral) as shown in Fig.



Two dimensional quadrilateral element

Solution:

The Jacobian matrix for a four noded element is given by,

$$[J] = \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

[illegible]

$N_1 = $ [illegible]

$N_2 = $ [illegible]

$N_3 = $ [illegible]

$N_4 = $ [illegible]

Prepared by:
[illegible]

Now, for a four node quadrilateral element the Jacobian matrix will become

$$[J]=\begin{bmatrix}\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi}\\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta}\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1-\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & -\frac{1+\eta}{4}\\ -\frac{1-\xi}{4} & -\frac{1+\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4}\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4\end{bmatrix}$$

Putting the values of ξ & η as 0.57735 and 0.57735 respectively,

$$\frac{\partial N_1}{\partial \xi}=-0.10566 \qquad \frac{\partial N_1}{\partial \eta}=-0.10566$$

$$\frac{\partial N_2}{\partial \xi}=0.10566 \qquad \frac{\partial N_2}{\partial \eta}=-0.39434$$

$$\frac{\partial N_3}{\partial \xi}=0.39434 \qquad \frac{\partial N_3}{\partial \eta}=0.39434$$

$$\frac{\partial N_4}{\partial \xi}=-0.39434 \qquad \frac{\partial N_4}{\partial \eta}=0.10566$$

Hence,

$$J_{11} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i = [illegible]$$

Similarly, $J_{12} = 0.0612$, $J_{21} = 0.2542$ and $J_{22} = 1.1062$

Hence,

$$J = \begin{bmatrix} 1.00000 & 0.0612 \\ 0.2542 & 1.1062 \end{bmatrix}$$

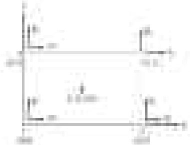
Thus, the inverse of the Jacobian matrix will become

$$[J]^{-1} = \begin{bmatrix} J^*_{11} & J^*_{12} \\ J^*_{21} & J^*_{22} \end{bmatrix} = \begin{bmatrix} [illegible] & [illegible] \\ [illegible] & [illegible] \end{bmatrix}$$

Hence strain displacement matrix is given by

$$[B] = \begin{bmatrix} J^*_{11} & J^*_{12} & 0 & 0 \\ 0 & 0 & J^*_{21} & J^*_{22} \\ J^*_{21} & J^*_{22} & J^*_{11} & J^*_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_4}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_4}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_4}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} [illegible] & [illegible] & 0 & 0 \\ 0 & 0 & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] & [illegible] \end{bmatrix} \begin{bmatrix} [illegible] & [illegible] & [illegible] & [illegible] & 0 & 0 & 0 & 0 \\ [illegible] & [illegible] & [illegible] & [illegible] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [illegible] & [illegible] & [illegible] & [illegible] \\ 0 & 0 & 0 & 0 & [illegible] & [illegible] & [illegible] & [illegible] \end{bmatrix}$$



$$\mathbf{J}=\begin{bmatrix}-\frac{1-\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & -\frac{1+\eta}{4}\\ -\frac{1-\xi}{4} & -\frac{1+\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4}\end{bmatrix}\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}J^{*}_{11} & J^{*}_{12} & 0 & 0\\ 0 & 0 & J^{*}_{21} & J^{*}_{22}\\ J^{*}_{21} & J^{*}_{22} & J^{*}_{11} & J^{*}_{12}\end{bmatrix}$$

$$\mathbf{A}=\frac{1}{\det \mathbf{J}}\begin{bmatrix}J_{22} & -J_{12} & 0 & 0\\ 0 & 0 & -J_{21} & J_{11}\\ -J_{21} & J_{11} & J_{22} & -J_{12}\end{bmatrix}$$

$$\mathbf{A}=\frac{1}{(1/2)}\begin{bmatrix}\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & \frac{1}{2} & 0\end{bmatrix}$$

$$\mathbf{B}=\mathbf{A}\mathbf{G} \qquad \boxed{\boldsymbol{\varepsilon}=\mathbf{B}\mathbf{q}}$$

$$\mathbf{G}=\frac{1}{4}\begin{bmatrix}-(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0\\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0\\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta)\\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi)\end{bmatrix}$$

Evaluating B at ξ = η = 0

$$B^0 = [illegible]$$

The stresses at ξ = η = 0 are now given by the product

$$\sigma^0 = DB^0q$$

$$D = \frac{30 \times 10^6}{(1 - 0.09)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Thus $\sigma^0 = [illegible]$

For the quadrilateral element shown in the figure, evaluate the stiffness matrix assuming E = [illegible] Pa and Poisson's ratio as 0.3, a unit thickness and a plane stress condition.

Gauss points

Points	ξ	η
P_1	-0.57735	-0.57735
P_2	0.57735	-0.57735
P_3	$+0.57735$	$+0.57735$
P_4	-0.57735	0.57735

[illegible]

For we consider (1)

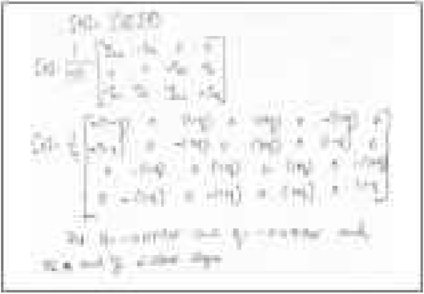
$\xi = -0.57735 \qquad \eta = -0.57735$

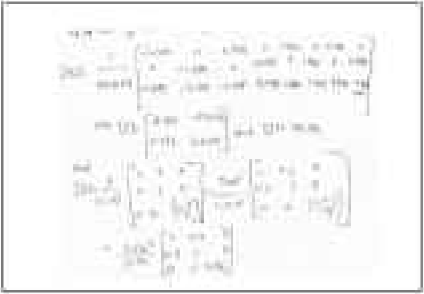
$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

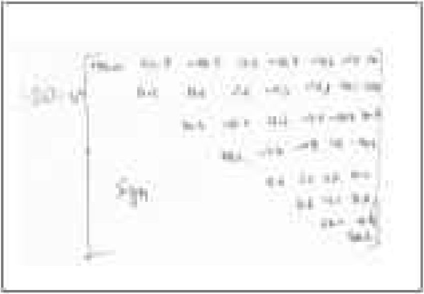
$$J_{12} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

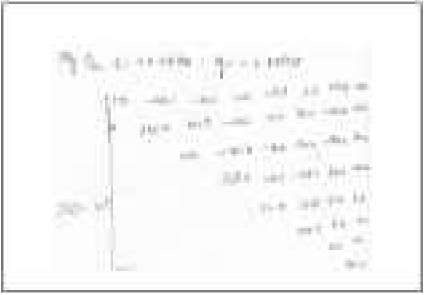
$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

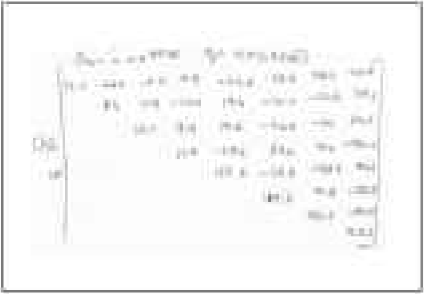
$$J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$$











[illegible]

[illegible]

Numerical Integration

Numerical Integration

In the finite element analysis, we face the problem of evaluating the following types of integrations in one, two and three dimensional cases respectively.

These are necessary to compute element stiffness and element load vector.

$$\int \phi(\xi)d\xi, \quad \int \phi(\xi,\eta)d\xi d\eta, \quad \int \phi(\xi,\eta,\zeta)d\xi d\eta d\zeta$$

Approximate solutions to such problems are possible using certain numerical techniques.

Several numerical techniques are available in mathematics for solving definite integration problems, including midpoint rule, trapezoidal rule, Simpson's 1/3rd rule, Simpson's 3/8th rule and Gauss Quadrature formula.

Among these, Gauss Quadrature technique is most useful as for solving problems in finite element method.

Gauss Quadrature for one-dimensional integrals

The concept of Gauss Quadrature is first illustrated in one dimension in the context of an integral in the form of

$$I = \int_{-1}^{1} \phi(\xi)\, d\xi \quad \text{from} \quad \int_{x_1}^{x_2} f(x)\, dx.$$

[illegible]

To transform from an arbitrary interval of $x_1 \le x \le x_2$ to an interval of $-1 \le \xi \le 1$, we need to change the integration function from $f(x)$ to $\phi(\xi)$ accordingly.

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2 = N_1 x_1 + N_2 x_2$$

$$\text{so for } \xi = -1,\; x = \frac{1-(-1)}{2} x_1 + \frac{1-1}{2} x_2 = x_1$$

$$\xi = -1, \quad x = x_1$$

$$\therefore I = \int_{x_1}^{x_2} f(x)\, dx = \int_{-1}^{1} \phi(\xi)\, d\xi$$

[illegible]

[illegible]

$$I=\int_{-1}^{1}\phi(\xi)\,d\xi \approx \sum_{i=1}^{n} W_i\,\phi(\xi_i) = W_1\phi(\xi_1)+W_2\phi(\xi_2)+\dots+W_n\phi(\xi_n)$$

One- Point Formula

Considering n = 1

$$\int_{-1}^{1}\phi(\xi)\,d\xi = W_1\,\phi(\xi_1)$$

Two-Point Formula

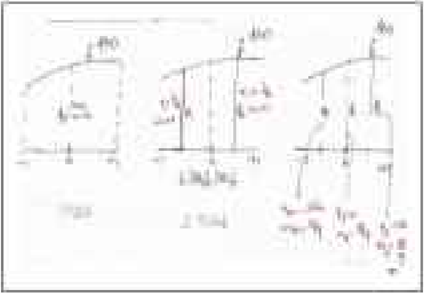
If we consider n = 2

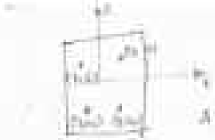
$$\int_{-1}^{1}\phi(\xi)\,d\xi = W_1\,\phi(\xi_1)+W_2\,\phi(\xi_2)$$

Gauss Quadrature for Two-Dimensional Integrals

$$I=\int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\eta)\,d\xi\,d\eta$$

$$I=\sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j\,\phi(\xi_i,\eta_j)$$



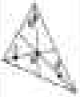


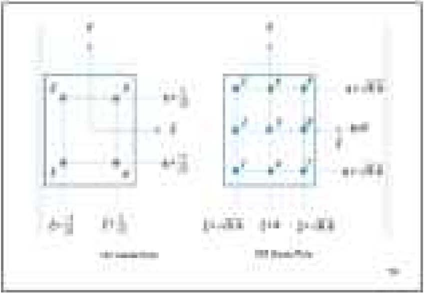
4 point Gauss Quadrature

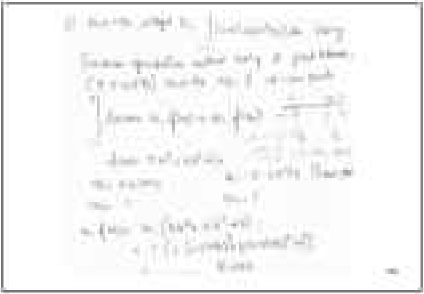
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j\, f(\xi_i,\eta_j)$$

Gauss points and corresponding weights

Number of Gauss points, n	Gauss Point Location, ξ_i	Weight, w_i
1	0.0	2.0
2	$\pm 0.5773502692 = \pm 1/\sqrt{3}$	1.0
3	0.0	0.8888888889 (=8/9)
	$\pm 0.7745966692 (= \pm\sqrt{0.6})$	0.5555555556 (=5/9)
4	±0.8611363116	0.3478548451
	±0.3399810436	0.6521451549







[illegible]

Check

[illegible]

... Result is verified

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

Evaluate $\int\int$ [illegible] $dx\,dy$

[illegible] gauss points

[illegible]

[illegible]

[illegible]

[illegible]

Hence $\iint$ [illegible]

Check $\iint$ [illegible]

[illegible]

verified

Axi-symmetric Element Formulation

Axisymmetric Analysis

At the same time, for some components treated as axisymmetric solids (or solids of revolution), even though all the three dimensions are comparatively large, they may be analysed using 2D techniques due to their axial symmetry.

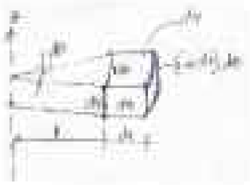
Examples:

- Pressure vessels
- Cylinders loaded by uniform internal and (or) external pressure
- Flywheels
- Turbine discs
- Circular footings resting on soil mass (soil)

In the axisymmetric analysis, the properties like stresses and deformations of the radial section of the body of revolution (with respect to symmetry) are identical.

Usually cylindrical coordinates (r, θ and z) are used in the analysis. Because of symmetry, all the deformations and stresses are independent of coordinate (θ). Thus the problem is reduced to 2D in the coordinates r and z.

Formulation



(ii) The radial strain is given by:

$$\epsilon_r = \frac{\left(u + \frac{\partial u}{\partial r} dr\right) - u}{dr} = \frac{\partial u}{\partial r}$$

(iii) The tangential strain (or circumferential strain):

$$\epsilon_\theta = \frac{(r+u)\,d\theta - r\,d\theta}{r\,d\theta} = \frac{u}{r}$$

[illegible]

(iv) The axial strain (or longitudinal strain):

$$\epsilon_z = \frac{\left(w + \frac{\partial w}{\partial z} dz\right) - w}{dz} = \frac{\partial w}{\partial z}$$

(v) The shear strain:

$$\gamma_{rz} = \frac{\left(u + \frac{\partial u}{\partial z} dz\right) - u}{dz} + \frac{\left(w + \frac{\partial w}{\partial r} dr\right) - w}{dr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

[illegible]

In matrix form

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix}$$

By stress

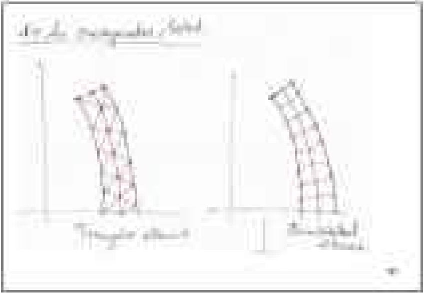
$$\{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \{\sigma_r \ \sigma_\theta \ \sigma_z \ \tau_{rz}\}^T$$

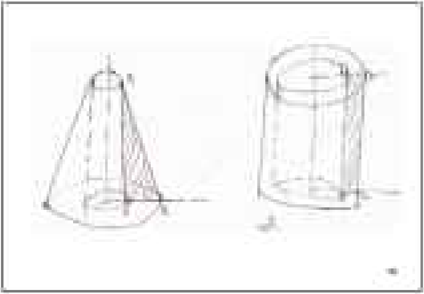
As per Hooke's law

$$\{\sigma\} = [D]\{\varepsilon\}$$

where

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & (1-\nu) & \nu & 0 \\ \nu & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$





In 2D displacement across thickness assumed to be zero. Other displacements (say u and v) are functions of x and y.

Similarly, in the case of axisymmetric solid, the circumferential displacement (in θ) is zero (i.e. v=0) and hence u and w displacements are the functions of r and z respectively.

u = f(r, z)
v = 0
w = g(r, z)

Definition of shape function for axisymmetric triangular element

Consider an axisymmetric triangular element with nodes 1, 2 and 3 as in the figure.

Let u_1, v_1, u_2, v_2 and u_3, v_3 are nodal displacements

[illegible]

CST element has [illegible] generalized coordinates for representing the displacement function at any point inside the element

[illegible]

Applying nodal conditions

[illegible]

Nodal displacements

[illegible]

Eqn 2 in matrix form

[illegible]

[illegible] in (5)

$u(x,y) = [illegible] \; \frac{1}{2A} \begin{bmatrix} [illegible] \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

$= \frac{1}{2A} [illegible] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

$= \left[\frac{[illegible]}{2A} \quad \frac{[illegible]}{2A} \quad \frac{[illegible]}{2A} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

$= [N_1 \; N_2 \; N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

$u = N_1u_1 + N_2u_2 + N_3u_3$ —— (6)

$v = N_1v_1 + N_2v_2 + N_3v_3$ —— (7)

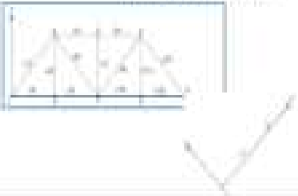
where [illegible] for [illegible] triangular element

$N_i = \frac{[illegible]}{2A}$ [illegible]

Miscellaneous Items

COORDINATE SYSTEMS

(i) Global coordinates
(ii) Local coordinates and
(iii) Natural coordinates.



Natural Coordinates

- A natural coordinate system is a coordinate system which permits the specification of a point within the element by a set of dimensionless numbers, whose magnitude never exceeds unity.

- It is obtained by assigning weightages to the nodal coordinates in defining the coordinate of any point inside the element.

- The use of natural coordinate system is advantageous in assembling element properties (stiffness matrices), since closed form integration formulae are available when the expressions are in natural coordinate system.

Natural Coordinates in one Dimension

Consider the two noded line element shown in Fig. let the natural coordinates of point P be (l_1, l_2) and the Cartesian coordinate be x. Node 1 and node 2 have the Cartesian coordinates x_1 and x_2.

Since natural coordinates are nothing but weightages to the nodal coordinates, total weightage at any point is unity i.e., $l_1 + l_2 = 1$ and also

$$L_1 + L_2 = 1$$

$$L_1 x_1 + L_2 x_2 = x$$

In matrix form:

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\begin{Bmatrix}L_1\\L_2\end{Bmatrix} = \begin{bmatrix}1 & 1\\ x_1 & x_2\end{bmatrix}^{-1}\begin{Bmatrix}1\\x\end{Bmatrix}$$

$$\begin{Bmatrix}L_1\\L_2\end{Bmatrix} = \frac{1}{x_2-x_1}\begin{bmatrix}x_2 & -x_1\\ -1 & 1\end{bmatrix}^{T}\begin{Bmatrix}1\\x\end{Bmatrix}$$

$$= \frac{1}{x_2-x_1}\begin{bmatrix}x_2 & -1\\ -x_1 & 1\end{bmatrix}\begin{Bmatrix}1\\x\end{Bmatrix} = \frac{1}{x_2-x_1}\begin{Bmatrix}x_2-x\\ -x_1+x\end{Bmatrix}$$

Noting that $x_2 - x_1 = l$, the length of the element

$$\begin{Bmatrix}L_1\\L_2\end{Bmatrix} = \begin{Bmatrix}\dfrac{x_2-x}{l}\\[2mm] \dfrac{x-x_1}{l}\end{Bmatrix}$$

The variation of L_1 and L_2

(a) variation of L_1

(b) variation of L_2

The standard closed form integration over entire length is

$$\int_0^l L_1^p \, L_2^q \, dx = \frac{p! \, q!}{(p+q+1)!} \, l$$

Integrate the following over the entire length l of the element

Here $p = 2$, $q = 0$

(i) $\int_0^l L_1^2 \, dx$

$$\int_0^l L_1^2 \, dx = \frac{2! \, 0!}{(2+0+1)!} \, l$$

$$= \frac{1}{3} \, l = \frac{l}{3} \quad \text{Answer}$$

(ii) $\int_0^l L_1^3 \, L_2 \, dx = \frac{3! \, 1!}{(3+1+1)!} \, l$

$$= \frac{3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \, l = \frac{l}{20} \quad \text{Answer}$$

Natural Coordinates in Three Dimension

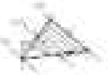
Tetrahedron coordinates

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{Bmatrix} \qquad \zeta_i = \frac{V_i}{V} \text{ for } i = 1, 2, 3 \text{ and } 4$$

$$\int_V \zeta_1^a \zeta_2^b \zeta_3^c \zeta_4^d \, dV = \frac{a!\, b!\, c!\, d!}{(3+a+b+c+d)!}\, 6V$$

Applications of the CST Element

Shape Functions



Nature of Finite Element Solution

Numerical Error

Types of Errors:

- Modeling Error (beams, plates … theories)
- Discretization Error (limits, geometries …)
- Numerical Error (in solving FE equations)

- Large difference in stiffness of different parts in FE model may cause ill-conditioning in FE equations. Hence giving results with large errors.
- Ill-conditioned systems of equations can lead to large changes in solution with small changes in input (right hand side vector)

Convergence of FE Solutions

As the mesh in an FE model is "refined" repeatedly, the FE solution will converge to the exact solution of the mathematical model of the problem (the model based on bar, beam, plane stress/strain, plate, shell, or 3-D elasticity theories or assumptions).

Types of Refinement:

h-refinement:	reduce the size of the element ("h" refers to the typical size of the elements);
p-refinement:	Increase the order of the polynomials on an element (linear to quadratic, etc.; "h" refers to the highest order in a polynomial);
r-refinement:	re-arrange the nodes in the mesh;
hp-refinement:	Combination of the h- and p-refinements (better results!).

Plate and Shell Elements

Plate Theory

- [illegible]
- [illegible]
- [illegible]

Applications

- [illegible]
- [illegible]
- [illegible]

[illegible]

[illegible]

Assumptions [illegible]:

A straight line along the normal to the mid-surface remains straight and normal to the deflected mid-surface after loading [illegible]

[illegible]

[illegible]

Thick Plate Theory (Mindlin Plate Theory)

If the thickness t of a plate is not "thin", e.g., t/L ≥ 1/10 (L = a characteristic dimension of the plate), then the thick plate theory by Mindlin should be applied. This theory accounts for the angle changes within a cross-section, that is,

$$\gamma_{xz} \neq 0, \quad \gamma_{yz} \neq 0$$

This means that a line which is normal to the mid surface before the deformation will not be so after the deformation.

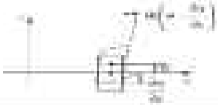


Plate Elements:

Kirchhoff Plate Elements:

Mindlin Plate Elements:

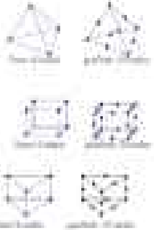
Shells and Shell Elements

Typical 3-D Solid Elements

Tetrahedron

Hexahedron (brick)

Penta



[illegible]

Axial Rod

To determine Governing Equation:

1. Equilibrium equation: $\frac{d\sigma_x}{dx} + B_x = 0$
2. Strain-displacement relation: $\varepsilon_x = \frac{du}{dx}$
3. Constitutive relation: $\sigma_x = E\varepsilon_x$

Using the above relations, we have

$$\sigma_x = E\frac{du}{dx}$$

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) + B_x = 0$$

[illegible]

Weighted Residual Method

[illegible]

$$= \int_{x_1}^{x_2} W_i\, EA \frac{d^2u}{dx^2}\, dx + \int_{x_1}^{x_2} W_i\, p\, dx$$

This is known as **strong** form [illegible]

$$= \left[W_i\, EA \frac{du}{dx}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{dW_i}{dx} EA \frac{du}{dx}\, dx + \int_{x_1}^{x_2} W_i\, p\, dx$$

This is known as **weak** form [illegible]

$$= W(x_2)P_2 - W(x_1)P_1 - \int \frac{dW}{dx} EA \frac{du}{dx}\, dx + \int W\, p\, dx$$

$EA \frac{du}{dx} = P$ — axial force in the section

Substituting [illegible] expression for $u(x)$

$u(x) = N_1(x)\, u_1 + N_2(x)\, u_2$

and substituting W as N_1 and N_2 [illegible]

$$0 = -\int_{x_1}^{x_2} \frac{dN_1}{dx} EA \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2\right) dx + \int_{x_1}^{x_2} N_1 p\, dx$$

$$0 = -\int_{x_1}^{x_2} \frac{dN_2}{dx} EA \left(\frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2\right) dx + \int_{x_1}^{x_2} N_2 p\, dx$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = -\begin{bmatrix} \int N_1' EA N_1'\, dx & \int N_1' EA N_2'\, dx \\ \int N_2' EA N_1'\, dx & \int N_2' EA N_2'\, dx \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} \int N_1 p\, dx \\ \int N_2 p\, dx \end{Bmatrix}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = -\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\mathbf{0 = -kd + F}$$

Thus, we obtain the discrete governing equation.

$$\mathbf{F = kd}$$

[illegible]

References

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[illegible]

All the best ...