Lecture Notes on

Aerostructures-II

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Objective

To study and analyse the behaviour of various thin-walled aerospace structural components under different load conditions.

Course Outcomes

CO1: Understand the behavior of aerospace structures.

CO2: Analysis of [illegible]

CO3: Evaluate the bending stresses and flexural shear flows in thin-walled sections.

CO4: Determine the torsional shear flow in thin-walled sections.

CO5: Illustrate the concepts of buckling for thin-walled sections.

Syllabus

Unit 1

Introduction to different aerospace structures, loads on an aircraft, characteristics of aircraft structures, basic structural members in aircraft structures and their functions; torsion in non-circular bars, Prandtl stress function, St. Venant warping function, membrane analogy, torsion in narrow rectangular sections.

Unit 2

Euler - Bernoulli and Timoshenko beam theories, bi-directional bending, bending and transverse shear stresses, bending stresses in narrow rectangular sections, general symmetric sections, and thin-walled sections. Flexural shear flows (SF), due to thin walled open sections; shear centre in open sections.

Unit 3

Torsional shear flows (TSF) in thin-walled open sections, TSF in thin walled closed sections (single and multiple cells) and warping in open and closed thin-walled sections. Combined flexural and torsional shear flows in thin-walled closed sections (single and multi cells) and shear center in closed sections. Buckling of one-dimensional sections and buckling of thin walled sections.

Text Book(s)

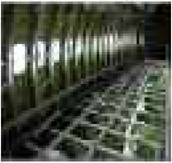
- C. T. Sun, "Mechanics of aircraft structures", 2nd Edition, John Wiley & Sons, New York, 2006.

Reference(s)

- Megson, T.H.G., "Aircraft Structures for Engineering Students", 4th edition Butterworth-Heinemann, USA, 2007.
  Peery, D.J., and Azar, J.J., "Aircraft Structures", 2nd edition, McGraw-Hill, New York, 1982.
- Bruhn, E.F., "Analysis and Design of Flight vehicles Structures", Tri-state offset company, USA, 1965.
- Rivello, R. M., "Theory and analysis of flight structures" McGraw-Hill, New York, 1969.

Introduction

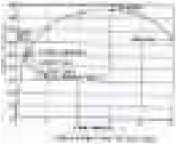
The main difference between aircraft structures and materials and civil engineering structures and materials lies in their weight.



Aircraft Structure

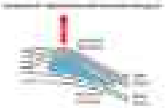
Civil Structure

The main driving force in aircraft structural design and aerospace material development is to reduce weight. In general, materials with high stiffness, high strength, and light weight are most suitable for aircraft applications.



Aircraft structures must be designed to ensure that every part of the material is used to its full capability. This requirement leads to the use of shell-like structures (monocoque construction) and stiffened shell structures (semi-monocoque construction).

The size and shape of an aircraft structural component are usually determined based on nonstructural considerations. For instance, the airfoil is chosen according to aerodynamic lift and drag characteristics.



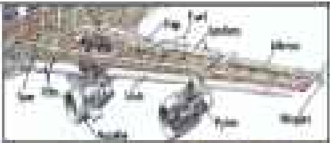
Because of their high stiffness/weight and strength/weight ratios, aluminum and titanium alloys have been the dominant aircraft structural materials for many decades. However, the recent advent of advanced fiber-reinforced composites has changed the outlook. Composites may now achieve weight savings of 30–40 percent over aluminum or titanium counterparts.

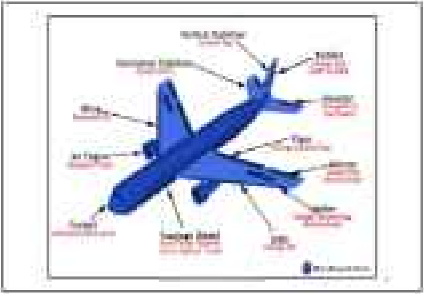
Materials	Specific ultimate strength	Specific yield point
	σ_u/ρ, km	σ_y/ρ, km
Titanium alloys	8.8-20	7.7-20
Steel	6.4-17	4.5-14
Aluminium alloys	11.3-26	6.6-24

BASIC STRUCTURAL ELEMENTS IN AIRCRAFT STRUCTURE

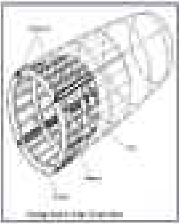
Major components of aircraft structures are assemblages of a number of basic structural elements, each of which is designed to take a specific type of load, such as axial, bending, or torsional load.

- Axial Member
- Shear Panel
- Bending Member (Beam)
- Torsion Member

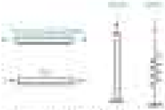





Boeing F/A-18F
SUPER HORNET



Axial Member

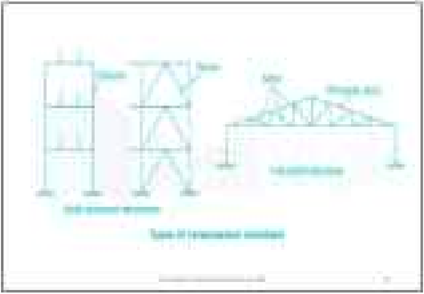


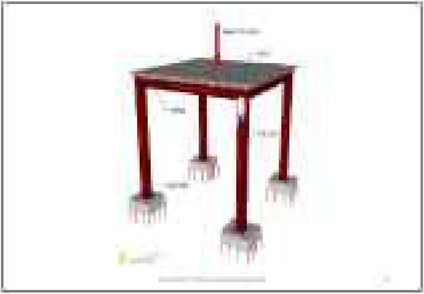
Axial members are used to carry (tensional or compressive) loads applied in the direction of the axial direction of the member.

The resulting stress is normal:

$$\sigma = E\varepsilon$$

where E and ε are the Young's modulus and normal strain, respectively, in the loading direction.



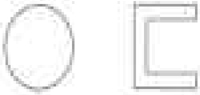


The total axial force F provided by the member is

$$F = A\sigma = EA\varepsilon$$

where A is the cross-sectional area of the member. The quantity EA is termed the axial stiffness of the member, which depends on the modulus of the material and the cross-sectional area of the member.

It is obvious that the axial stiffness of axial members cannot be increased (or decreased) by changing the shape of the cross-section. In other words, a circular rod and a channel can carry the same axial load as long as they have the same cross-sectional area.



[illegible] [illegible]

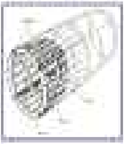
(a) Circular rod. (b) Channel.

Axial members are usually slender and are susceptible to buckling failure when subjected to compression.

Buckling strength can be increased by increasing the bending stiffness and by decreasing the length of the buckle mode.

For buckling, the channel section is better since it has higher bending stiffness than the circular section.

However, because of the slenderness of most axial members used in aircraft (such as stringers), the bending stiffness of these members is usually very small and is not sufficient to achieve the necessary buckling strength. In practice, the buckling strength of axial members is enhanced by providing lateral supports along the length of the member with more rigid ribs (in wings) and frames (in fuselages).



Shear Panel

A shear panel is a thin sheet of material used to carry in-plane shear load

Shear panels are critical components in aircraft structures, primarily used in wings and fuselages to resist shear forces and distribute loads effectively. They are typically composed of thin plates, like the skins of fuselage skin, reinforced by stiffeners (stringers and frames). These panels play a vital role in preventing buckling and maintaining overall structural integrity under various flight loads.

Function and Importance

Load Distribution:

Shear panels transfer shear loads from the skin to the stiffeners and the frames, ensuring that the loads are distributed throughout the structure.

Structural Integrity:

They contribute to the overall stiffness of the aircraft, resisting bending, twisting, and other deformations.

Prevention of Buckling:

[illegible]

Fatigue Life:

By distributing loads and reducing stress concentrations, shear panels can increase the fatigue life of the aircraft structure.

Impact Resistance:

They help withstand impact loads from foreign object debris or ground handling.

Applications:

- Wing

- Fuselage

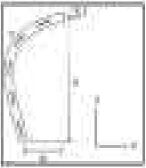
- Stabilizers

Consider a shear panel under uniform shear stress τ as shown in Fig. 1.2. The total shear force in the x-direction provided by the panel is given by

$$V_x = \tau \cdot t \cdot a = G\gamma t a$$

where G is the shear modulus, and γ is the shear strain. Thus, for a flat panel, the shear force V_x is proportional to its thickness and the lateral dimension a.

For a curved panel under a state of constant shear stress τ (see Fig. 1.5), the resulting shear force of the shear stress on the thin-walled section may be decomposed into a horizontal component V_x and a vertical component V_y as



$$V_x = \tau \cdot t \cdot a = Gat\gamma$$

$$V_y = \tau \cdot t \cdot b = Gbt\gamma$$

Thus, the components of the resultant force of the shear stress have the relation

$$\frac{a}{b} = \frac{V_x}{V_y}$$

Since the contour shape of the panel is irrelevant, it is obvious that a flat panel is more cost effective in resisting shear forces.

Bending Member(Beam)

In aircraft, bending members are structural components designed to withstand bending loads, which are forces that cause a member to rotate or bend. These members are crucial for maintaining the structural integrity of the aircraft, particularly during flight when aerodynamic forces exert significant bending moments on the wings and fuselage.

Key Bending Members and Their Functions:

-Wings:

The wings are primary bending members, with spars acting as the main structural elements that [illegible] from the fuselage to the wingtip. These spars experience significant bending due to lift forces acting on the wing.

-Fuselage:

The fuselage, the main body of the aircraft, also experiences bending loads, particularly from the wings and empennage (tail) sections.

-Tail Surfaces:

Control surfaces like the elevators and ailerons, as well as the empennage (the vertical and horizontal stabilizers), also act as bending members. The ribs and spars [illegible] withstand bending forces.

A structural member that can carry bending moment is called a beam. A beam can also act as an axial member carrying longitudinal tension and compression. According to simple beam theory, bending moment M is related to beam deflection w as:

$$M = -EI\frac{d^2w}{dx^2}$$

where EI is the bending stiffness of the beam. The axial moment of inertia I depends on the geometry of the cross-section.

Except for pure moment bending, a beam is designed to carry both bending moments and transverse shear forces as the latter usually produce the former.

For a beam of a large span/depth ratio, the bending stress is usually more critical than the transverse shear stress.

It is easy to see that the maximum bending moment and bending stress occur at the fixed end of the cantilever beam. We have

$$\sigma_{max} = \frac{M_{max}(h/2)}{I} = \frac{VL(h/2)}{bh^3/12} = \frac{6VL}{bh^2}$$

The transverse shear stress distribution is parabolic over the beam depth with maximum value occurring at the neutral plane, i.e.,

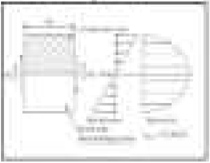
$$\tau_{max} = \frac{3}{2}\frac{V}{bh}$$

Form the ratio

$$\frac{\sigma_{max}}{\tau_{max}} = \frac{4L}{h}$$

It is evident that bending stress plays a more dominant role than transverse shear stress if the span-to-depth ratio is large (as in wing structures). For such beams, attention is focused on optimizing the cross-section to increase bending stiffness.

In the elastic range, bending stress distribution over depth is linear with maximum values at the farthest positions from the neutral axis. The material near the neutral axis is underutilized. Thus, the beam with a rectangular cross section is not an efficient bending member.



In order to utilize the material to its full capacity, material in a beam must be located as far as possible from the neutral axis. An example is the wide [illegible] Although the bending stress distribution is still linear over the depth, the bending [illegible] middle distribution is concentrated at the two flanges because $b \gg t_w$. For simplicity, the small contribution of the vertical web in bending can be neglected.

The transverse shear stress distribution in the wide flange beam is shown in Fig. The vertical web is seen to carry essentially all the transverse shear load; its variation over the web is small and can be practically assumed to be constant. For all practical purposes, the wide flange beam can be regarded as two axial members (flanges) connected by a thin shear panel.

Torsion Member

Torque is an important form of load in aircraft structures. In a structural member, torque is formed by shear stresses acting in the plane of the cross section.

Hollow cylinder subjected to a torque

Consider a hollow cylinder subjected to a torque, T

1.6. The torque induced shear stress τ is linearly distributed along the radial direction. The torque is related to the twist angle θ per unit length as

$$T = GJ\theta \qquad \boxed{\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}}$$

where GJ is the torsional stiffness and, for hollow cylinders;

$$J = \frac{1}{2}\pi(b^4 - a^4)$$

Now that the material near the inner circle in a thick-walled cylinder is understressed, it is obvious that a thin-walled tube would be more efficient for torques than a solid cylinder or a thick-walled hollow cylinder. Figure

Cross-sections of (a) a solid cylinder and (b) a tube

Both cylinder and tube having the same amount of material. It is easy to show that the torsional stiffness of the tube is about 150 times that of solid cylinder. Thin-walled structures can be made into a very efficient torsional member.

BASIC STRUCTURAL ELEMENTS IN AIRCRAFT STRUCTURE

Fixed wing aircraft parts

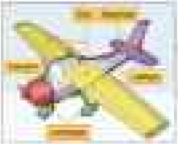
Various fuselage configurations

Various wing configurations

monospar | multispar type | box beam type

Wing Structure

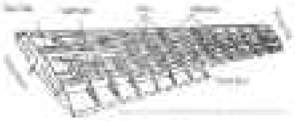
The main function of the wing is to pick up the air loads and transmit them to the fuselage. The wing cross-section takes the shape of an airfoil, which is designed based on aerodynamic considerations. The wing as a whole performs the combined function of a beam and a torsion member.



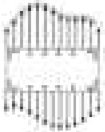
wing consists of axial members in stringers. Bending moment in spars and shear panels in the cover skin and webs of spars. The spar is a heavy beam running spanwise to take transverse shear loads and spanwise bending. It is usually composed of a thin shear panel (the web) with a heavy cap or flange at the top and bottom to take bending.

Typical wing cross sections (for subsonic aircraft)

Wing ribs are planar structures capable of carrying in-plane loads. They are placed chordwise along the wing span. Besides serving as load redistributors, ribs also hold the skin-stringer to the designed contour shape. Ribs reduce the effective buckling length of the stringers (or the stringer-skin system) and thus increase their compressive load capability. Figure 1.11 shows a typical rib construction. Note that the rib is supported by spars (or spar).



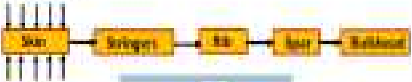
The cover skin of the wing together with the spar webs form an efficient torsion member. For subsonic airplanes, the skin is relatively thin and may be designed to undergo postbuckling. Thus, the thin skin can be assumed to make no contribution to bending of the wing box, and the bending moment is taken by spars and stringers.

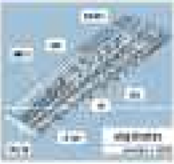


Supersonic airfoils are relatively thin compared with subsonic airfoils. To withstand high surface air loads and to provide additional bending capability of the wing box structure, thicker skins are often necessary. In addition, to increase structural efficiency, stiffeners can be manufactured (either by forging or machining) as integral parts of the skin.

Wing cross-sections with integrally stiffened skin.

Load Transfer

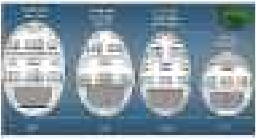




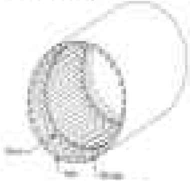
Fuselage

Unlike the wing, which is subjected to large distributed air loads, the fuselage is subjected to relatively small air loads. The primary loads on the fuselage include large concentrated forces from wing reactions, landing gear reactions, and pay loads.

For airplanes carrying passengers, the fuselage must also withstand internal pressure. Because of internal pressure, the fuselage often has an efficient circular cross-section. The fuselage structure is a semimonocoque construction consisting of a thin shell stiffened by longitudinal axial elements (stringers and longerons) supported by many transverse frames or rings along its length, see Fig.



The fuselage skin carries the shear stresses produced by torques and transverse forces. It also bears the hoop stresses produced by internal pressure. The stringers carry bending moments and axial forces. They also stabilize the thin fuselage skin.



Aircraft Materials

Traditional metallic materials used in aircraft structures are aluminum, titanium, and steel alloys. In the past three decades, applications of advanced fiber composites have rapidly gained momentum. To date, some new commercial jets, such as the Boeing 787, already contain composite materials up to 50% of their structural weight.

Selection of aircraft materials depends on many considerations that can, in general, be categorized as cost and structural performance.

Cost includes **initial material cost, manufacturing cost, and maintenance cost.**

The key material properties that are pertinent to maintenance cost and structural performance are as follows:

- Density (weight)
- Stiffness (Young's modulus)
- Strength (ultimate and yield strengths)
- Durability (fatigue)
- Damage tolerance (fracture toughness and crack growth)
- Corrosion.

Steel Alloys

Among the three metallic materials, steel alloys have highest densities, and are used only where high strength and high yield stress are critical.
Examples include landing gear units and highly loaded fittings.
The high strength steel alloy 300M is commonly used for landing gear components. This steel alloy has a strength of 1.9 GPa and a yield stress of 1.5 GPa.

Besides being heavy, steel alloys are generally poor in corrosion resistance; components made of these alloys need to be plated for corrosion protection.

Material Property	E GPa (psi)	ν	σ_u MPa (ksi)	σ_y MPa (ksi)	ρ g/cm³ (lb/in³)
[illegible]					
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]					
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Aluminum Alloys

Aluminum alloys have played a dominant role in aircraft structures for many decades. They offer good mechanical properties with low weight.

Among the aluminum alloys, the 2024 and 7075 alloys are perhaps the most [illegible].

The 2024 alloys (2024-T3, T42) have excellent fracture toughness and slow crack growth rate as well as good fatigue life.

The 2024-T3 is used in the fuselage and lower wing skins, which are prone to fatigue due to applications of cyclic tensile stresses.

For the upper wing skins, which are subjected to compressive stresses, fatigue is less of a problem, and 7075-T6 is used.

The recently developed aluminum lithium alloys offer improved properties over conventional aluminum alloys. They are about 10% stiffer and 10% lighter and have superior fatigue performance.

Titanium Alloys

Titanium such as Ti–6Al–4V with a density of 4.5 g/cm3 is lighter than steel (7.8 g/cm3) but heavier than aluminum (2.7 g/cm3).

Its ultimate and yield stresses are almost double those of aluminum 7075-T6. Its corrosion resistance in general is superior to both steel and aluminum alloys.

While aluminum is usually not for applications above 350 °F, titanium, on the other hand, can be used continuously up to 1000 °F.

Titanium is difficult to machine and thus the cost of machining titanium parts is high. Near net shape forming is an economic way to manufacture titanium parts.

Despite its high cost, titanium has found increasing use in military aircraft. For instance, the F-15 contained 26% (structural weight) titanium.

Fiber-Reinforced Composites

- Fiber composites are stiff, strong, and light and are thus most suitable for aircraft structures.
- They are often used in the form of laminates that consist of a number of unidirectional laminae with different fiber orientations to provide multidirectional load capability. Composite laminates have excellent fatigue life, damage tolerance, and corrosion resistance.
- Laminate construction offers the possibility of tailoring fiber orientations to achieve optimal structural performance of the composite structure.

The range of service temperatures of a composite is often determined by its matrix material.

PMCs are usually for lower temperature (less than 300 °F) applications, and CMCs are intended for applications in hot (higher than 1500 °F) environments, such as jet engines.

Introduction to Elasticity

The primary design requirement of aircraft structural components is to ensure that the designed structures do not fail or excessively deform due to various ground and air loads.

From mechanics point of view, a basic structural design process starts by establishing the relation between external forces with the internal resultants.

The external loads are typically defined in the form of prescribed displacements, traction/stress vector, point forces, moments, or distributed, normal, or shear forces.

In a deformable body, the internal resultant forces and moments deform the body. The states of deformation are expressed in terms of strain components.

Depending on the external loading type and geometry of the structure, the applied loading can lead to uniform or nonuniform internal stress distributions.

Various branches of mechanics are simultaneously utilized to analytically perform a basic structural design process.

Theory of elasticity is a branch of mechanics that describes the elastic relation between stress and strain of a deformable body in equilibrium.

The mathematical formulations allow solving problems including uniform and nonuniform distributions of stresses.

In principle, theory of elasticity requires basic understanding of displacement, forces, stress, and strains in terms of mathematical equations.

Statics is the branch of mechanics concerned with the analysis of loads (force, moment) on physical systems in static equilibrium.

The foundation extension of statics is essentially an of Newton's first law.

Using statics, it is possible to obtain internal forces and moments on a deformable body subjected to external forces and moments.

However, it is not possible to relate forces with deformation using statics. To obtain such relation, the concepts from mechanics of solids is utilized.

Solid mechanics is the branch of mechanics that describes the relation between stress (obtained from the internal forces/moments of a body) and strain (obtained from the associated deformation in the body) of a body in static equilibrium.

The relation between stress and strain is established using Hooke's law where properties of materials such as Young's modulus (E) and Poisson's ratio (ν) appear as the "bridge" between stress and strain.

Therefore, properties of materials must be known if one wants to employ the concepts of solid mechanics.

The governing equations in the theory of elasticity are built upon displacement based formulation or stress based formulations.

In the displacement formulation, the displacement vectors are obtained first. Then, strain displacement relations are established in terms of differential equations.

Using the generalized Hooke's law the stress components are obtained.

In the **stress formulation**, stresses are obtained first, and then strains are computed using the stress-strain relations. Finally displacements are obtained using the strain displacement relations.

Displacement formulation

stress formulation

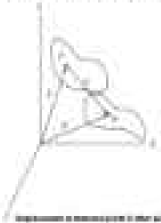
The theory of elasticity plays an important role for stress analysis in aircraft structural components.

However, such theory is only applicable to simple structural components, and hence, not suitable for analyzing real aerospace structures.

Advanced, experimentally validated computational methods such as finite element analysis are generally used for this purpose.

The theory of elasticity is often employed as a guideline for advanced computational methods.

CONCEPT OF DISPLACEMENT



Displacement of a material point P after deformation.

Consider a material point P at the position $P(x, y, z)$ before deformation. After deformation, P moves to a new position $P'(x', y', z')$.

The change of position during deformation, which is measured in terms of the displacement vector **u**, has three components u, v and w in the x, y and z directions, respectively.

The new location of the point (x, y, z) after deformation is given by

$$x' = x + u \quad \text{or} \quad u = x' - x$$
$$y' = y + v \qquad v = y' - y$$
$$z' = z + w \qquad w = z' - z$$

Thus, the deformed configuration is uniquely defined if the displacement components u, v and w are given everywhere in the body of interest.

Consider an axial member (i.e. a one-dimensional, 1-D, body) of original length L0. Assume the axial strain to be uniform in the member. Then the axial strain everywhere in the member is calculated by

$$\varepsilon = \frac{\Delta L}{L_0}$$

where ΔL is the total elongation of the member. The elongation ΔL can be regarded as the difference in displacement $u_L = u(L)$ at the right end and $u_0 = u(0)$ at the left end, i.e.

$$\Delta L = u_1 - u_0$$

The function $u(x) = u_0 + (x/L)(u_1 - u_0)$ gives the axial displacement at any point x in the axial member.

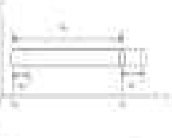
To determine the strain at a point, a small segment $AB = \Delta x$ must be considered.

Consider two points x_0 and $x_0 + \Delta x$ that are separated by a small distance Δx.

Let the displacements of these two points be:

$$u_0 = u(x_0)$$

$$u_1 = u(x_0 + \Delta x)$$



The difference in displacement between these two points is

$$\Delta u = u_1 - u_0 = u(x_0 + \Delta x) - u(x_0)$$

which can also be regarded as the elongation of the material between these two points.

The axial strain in this segment (or at point x_0) is defined as

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}$$

Thus, axial strain can be obtained from the derivative of the displacement function.

If a rod is subjected to a uniform tension and $\varepsilon = \varepsilon_0$ = constant, then

$$\frac{du}{dx} = \varepsilon_0$$

Integrate the equation above to obtain

$$u = \varepsilon_0(x - x_0) + u_0$$

Let $u(x_0) = u_0$, then, from the equation above, $C = u(x_0) = u_0$, and the displacement function is given by

$$u = \varepsilon_0(x - x_0) + u_0$$

STRAIN

Consider two points P and Q in a solid body. The coordinates of P and Q are (x, y, z) and (x + Δx, y, z), respectively.

The distance between the two points before deformation is Δx

After deformation, let the displacement of P in the x-direction be $u = u(x, y, z)$ and of Q be $u' = u(x + \Delta x, y, z)$.

The new distance between these two points (P' and Q') in the x-direction after deformation is

$$(x + \Delta x + u') - (x + u) = \Delta x + \Delta u$$

where $\Delta u = u' - u$ is the change of length in the x-direction in a material connecting P and Q after deformation.

[illegible]

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$$

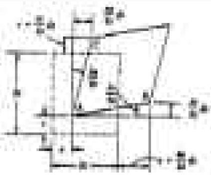
This is the x-component of the normal strain, which measures the deformation in the x-direction at a point (x, y, z).

Similarly, the y and z components of the normal strain at the point are given by

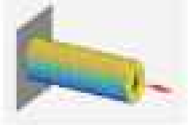
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Comparing the strain component ε_{xx} with the strain in the 1-D case (or in an axial member), we may interpret ε_{xx} as the elongation per unit length of an "infinitesimal" axial element of the material at a point (x, y, z) in the x direction.

Similar interpretations can be given to εyy and εzz.



shear strain is $\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$



The three normal strain components are not sufficient to describe a general state of deformation in a 3-D body. Additional shear strain components are needed to describe the directional deformation.

For simplicity let us consider a 2-D case. Let P, Q, R be three neighboring points all lying on the x - y plane as shown in Figure.

Let P', Q', and R' be the corresponding positions after deformation.

For no loss of generality, we assume that P does not move. In addition, we assume that the infinitesimal elements dx and dy do not experience any elongation.

Thus, the positions of P' and Q' are given by

$$Q' : (x + \Delta x, y + \Delta v)$$
$$R' : (x + \Delta u, y + \Delta y)$$

For Q', the displacement increment Δv is

$$\Delta v = \underbrace{v(x + \Delta x, y)}_{\text{displacement at } Q} - \underbrace{v(x, y)}_{\text{displacement at } P}$$

Similarly for R', the displacement increment Δu can be written as

$$\Delta u = u(x, y + \Delta y) - u(x, y)$$

The rotations θ_1 and θ_2 of elements $\overline{PQ}$ and $\overline{PR}$ are assumed to be small and are given by:

$$\theta_1 = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{\partial v}{\partial x} \quad \text{and} \quad \theta_2 = \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y} = \frac{\partial u}{\partial y}$$

The total change of angle between $\overline{PQ}$ and $\overline{PR}$ after deformation is defined as the shear strain component in the x-y plane:

$$\gamma_{xy} = \gamma_{yx} = \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Similar shear strain components in the y-z plane and z-x plane are defined as

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \gamma_{zx} = \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Thus a general state of deformation at a point in a solid is described by three normal strain components $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$ and three shear strain components $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

STRESS

For an axial member, the force is always parallel to the member, and the stress is defined as

$$\sigma = \frac{P}{A}$$

The concept of stress can easily be extended to 3-D bodies subjected to loads applied in arbitrary directions.

Consider an infinitesimal plane surface of area ΔA with a unit normal vector $\mathbf{n}$. The total resultant force acting on this area is $\Delta \mathbf{F}$.

The stress vector is defined as $\quad \mathbf{\tau} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$

Consider the special plane surface with the unit normal vector parallel to the x-axis.
On this face, the stress vector T has three components, which are denoted by σ_{xx}, τ_{xy}, τ_{xz} as shown in Fig. Similarly on the y and z faces the force intensities are given by the components of the respective stress vectors as σ_{yy}, τ_{yx}, τ_{yz}, σ_{zz}, τ_{zx}, τ_{zy}

EQUATIONS OF EQUILIBRIUM IN A NONUNIFORM STRESS FIELD

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

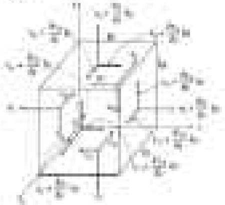
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

EQUATIONS OF EQUILIBRIUM



Stresses on the faces of an element at a Point in an Elastic Body

Taking moments about an axis through the centre of the element parallel to the z axis

$$\tau_{xy}(dydz)\frac{dx}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}dx\right)dydz\frac{dx}{2} - \tau_{yx}(dxdz)\frac{dy}{2} - \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)dxdz\frac{dy}{2} = 0$$

which simplifies to

$$\tau_{xy}dydzdx + \frac{\partial \tau_{xy}}{\partial x}dydz\frac{(dx)^2}{2} - \tau_{yx}dxdzdy - \frac{\partial \tau_{yx}}{\partial y}dxdz\frac{(dy)^2}{2} = 0$$

dividing through by dxdydz and taking the limit as dx and dy approach zero.

Similarly

$$\left.\begin{array}{l} \tau_{xy} = \tau_{yx} \\ \tau_{yz} = \tau_{zy} \\ \tau_{xz} = \tau_{zx} \end{array}\right\}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

PLANE STRESS

[illegible]

[illegible]

For isotropic materials and assuming

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\gamma_{xz} = \gamma_{yz} = 0.$$

$$\{\sigma\} = [D]\,\{\varepsilon\}$$

Where $[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

PLANE STRAIN

Plane strain is defined to be a state of strain in which the strain normal to the x-y plane, ε_z, and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$$\tau_{xz} = \tau_{yz} = 0$$

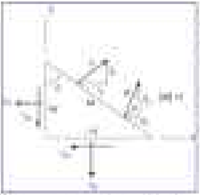
$$\{\sigma\} = [D]\{\varepsilon\}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0$$

$$\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0$$

STRESS VECTOR AND STRESS COMPONENTS RELATIONS



Two-dimensional state of stress on a wedge element

Consider a two dimensional state of stress in which $\sigma_z = 0$, $\tau_{xz} = 0$, $\tau_{yz} = 0$ and other stress components exists.

Consider the wedge shown in figure. The unit normal vector to the inclined surface is n and the stress vector (force per unit area) acting on this surface is t.

From the equilibrium equations $\Sigma F_x = 0$ and $\Sigma F_y = 0$ for the wedge we obtain:

$$t_x \, \Delta s = \sigma_{xx} \, \Delta y + \tau_{xy} \, \Delta x$$

$$t_y \, \Delta s = \tau_{xy} \, \Delta y + \sigma_{yy} \, \Delta x$$

(1.26)

By noting

$$\frac{\Delta y}{\Delta s} = \cos\theta = n_x \qquad \frac{\Delta x}{\Delta s} = \sin\theta = n_y$$

Eq. (1.26) can be expressed in the form

$$t_x = \sigma_{xx} n_x + \tau_{xy} n_y$$

$$t_y = \tau_{xy} n_x + \sigma_{yy} n_y$$

(1.28)

Equation (1.28) can be expressed in matrix form as

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

Using the same method, one can easily derive the equations for the three dimensional case with the result:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}$$

$$[\sigma]\{n\} = \{t\}$$

Assume that the uniform state of stress in the wedge shaped body shown in Figure is

$$[\sigma] = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{MPa}$$

What is the stress vector on the inclined face?

The unit vector $\mathbf{n}$ normal to the inclined face is given by

$$n_x = n_y = n_z = \frac{1}{\sqrt{3}}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}$$

[illegible]

[illegible]

PRINCIPAL STRESS

If we are interested in finding surfaces for which t is parallel to n, i.e.

$$[\sigma]\{n\} = \{t\}$$

$$\{t\} = \sigma\{n\}$$

where σ is a scalar. Then yields a typical eigenvalue problem

$$([\sigma] - \sigma[I])\{n\} = 0$$

where [I] is the identity matrix. In order for to have a nontrivial solution for {n}, we require that

[illegible]

$$|[\sigma] - \sigma[I]| = 0$$

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{vmatrix} = 0 \qquad (2.11)$$

Expanding the determinant of Eq.(2.11) yields a cubic equation for σ. Since [σ] is real and symmetric, there are three real roots, $\sigma = \sigma_1, \sigma_2, \sigma_3$ (we urge book on linear algebra or matrix theory for the proof). The corresponding eigenvectors $\{n^{(1)}\}, \{n^{(2)}\}, \{n^{(3)}\}$ can be shown to be mutually orthogonal. These three directions are called **principal directions** of stress, and σ_1, σ_2, and σ_3 are the corresponding **principal stresses**.

Given the stress tensor acting at point, determine Principal stresses and Principal planes.

$$\sigma = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Characteristic equation $|\sigma - \lambda I| = 0$

$$|\sigma - \lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$ **Eigen values λ = 0, 3, 15**

Principal Planes determination

$(a - \lambda I)\{e_j\} = 0$

[illegible]

Solving the above equations we will get [illegible]

Similarly by putting values of λ = [illegible] and [illegible]

other Eigen vectors can be determined

[illegible]

2D Principal Stresses

$$\sigma_{max}, \sigma_{min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Isotropic yield criteria

Maximum Principal Stress Theory $\sigma_1 \le \sigma_y$

Maximum Principal Strain Theory $\sigma_1 - \nu(\sigma_2 + \sigma_3) \le \sigma_y$

Maximum Shear Stress Theory (Tresca Theory) $\tau = \dfrac{\sigma_1 - \sigma_3}{2} \le \tau_y$

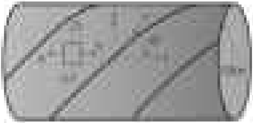
Total Strain Energy Theory $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) \le \sigma_y^2$

Distortion Energy Theory (von Mises Theory) $\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \le \sigma_y^2$

STRESS TRANSFORMATION

$$\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau'_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \frac{1}{2}\sin 2\theta & \cos 2\theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

As shown in Figure 3.12, the cylindrical pressure vessel has an inner radius of 1.25 m and a wall thickness of 18 mm. It is made from steel plates that are welded along the 45° seam. Determine the normal and shear stress components acting on the seam if the vessel is subjected to an internal pressure of [illegible] MPa.



Considering the x- and y-axis is along the longitudinal and circumferential direction of the pressure vessel, the stresses due to internal pressure can be obtained as:

$$\sigma_1 = \sigma_x = \frac{pr}{2t} = 0.5\ \text{MPa and } \sigma_2 = \sigma_y = \frac{pr}{t} = 1\ \text{MPa}$$

The x' axis is along the seam, as shown in Figure 3.12. To obtain the normal (σ'_x) and shear stress (τ'_{xy}) developed across the seam, the **coordinate transformation** formula described in section three (Eq. 3.XX) can be utilized:

$$\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau'_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2(45^\circ) & \sin^2(45^\circ) & \sin(2\cdot 45^\circ) \\ \sin^2(45^\circ) & \cos^2(45^\circ) & -\sin(2\cdot 45^\circ) \\ -\frac{1}{2}\sin(2\cdot 45^\circ) & \frac{1}{2}\sin(2\cdot 45^\circ) & \cos(2\cdot 45^\circ) \end{bmatrix} \begin{Bmatrix} 0.5 \\ 1 \\ 0 \end{Bmatrix} \text{MPa} = \begin{Bmatrix} 0.75 \\ 0.75 \\ 0.25 \end{Bmatrix} \text{MPa}$$

As result, the normal and shear stress components across the seam are 0.75 MPa (σ'_x) and 0.25 MPa (τ'_{xy}), respectively.

LINEAR STRESS–STRAIN RELATIONS

six independent stress components ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}$) to describe the load carried by a three-dimensional solid at a point, and six independent strain components ($\varepsilon_{x}, \varepsilon_{y}, \varepsilon_{z}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$) to describe the deformation at a point.

In general, deformations are produced by loads, strain components are related to stress components. These stress-strain relations are used to characterize the stiffness of a material.

$$\sigma_{xx} = E_x \varepsilon_{xx}$$

If the material is isotropic, i.e., its mechanical properties are not direction dependent, all the ν coefficients equal and

$$E_x = E_y = E_z = E$$

$$\nu_{xy} = \nu_{yx} = \nu_{xz} = \nu_{zx} = \nu_{yz} = \nu_{zy} = \nu$$

$$G_{xy} = G_{xz} = G_{yz} = G$$

Three dimensional stress-strain relations

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = [S_{ij}] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$$\{\varepsilon\} = [S]\{\sigma\}$$

$$\{\sigma\} = [C]\{\varepsilon\}$$

Orthotropic Materials

$$
\left[s_{ij}\right] = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix}
$$

$$
s_{11} = \frac{1}{E_1} \quad s_{12} = -\frac{\nu_{21}}{E_2} \quad s_{13} = -\frac{\nu_{31}}{E_3}
$$

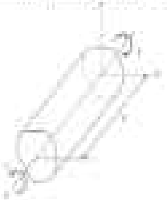
$$
s_{21} = -\frac{\nu_{12}}{E_1} \quad s_{22} = \frac{1}{E_2} \quad s_{23} = -\frac{\nu_{32}}{E_3}
$$

$$
s_{31} = -\frac{\nu_{13}}{E_1} \quad s_{32} = -\frac{\nu_{23}}{E_2} \quad s_{33} = \frac{1}{E_3}
$$

$$
s_{44} = \frac{1}{G_{23}} \quad s_{55} = \frac{1}{G_{13}} \quad s_{66} = \frac{1}{G_{12}}
$$

Torsion

Torque is a common form of load in aircraft structures. A torque is a moment or couple that has the unit N · m.



The difference between a torque and a bending moment is that a torque acts about the longitudinal axis of a shaft as illustrated in Figure, whereas a bending moment acts about an axis that is perpendicular to the longitudinal axis of the shaft (beam).

The effect of torque is of major concern in the design of many aircraft structural components such as wing, fuselage, horizontal, vertical stabilizers etc.

In analyzing torsion on a structural member the first step is to determine the relation between the applied torque and the internal deformation and stress fields.

The geometry of the section plays a significant role in determining how a torsion problem can be analyzed.
For example, torsion in a prismatic shaft of isotropic and linearly elastic solids can be treated using the concepts of solid mechanics.
Using solid mechanics approach, only shafts of circular solid and hollow sections can be analyzed. Other types of sections are unable to satisfy the assumptions involved in deriving the deformation and stress fields:

- Plane sections of the shaft remain plane and circular after deformation produced by application of the torque.
- Diameters in plane sections remain straight after deformation.

These assumptions lead to the result that shear stress (and thus shear strain) is a linear function of the radial distance from the point of interest to the center of the section.

Moreover, these assumptions imply that plane sections of the shaft rotate as rigid bodies during deformation without in-plane deformations or out-of-plane displacements (i.e. warping).

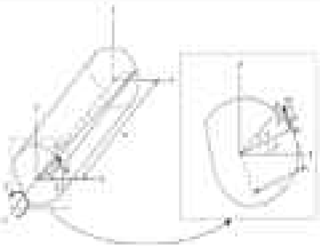
Unfortunately, these assumptions are not valid in shafts of noncircular sections, for which different formulations are required.

TORSION OF UNIFORM BARS WITH ARBITRARY CROSS-SECTION

There are two classical approaches to solving the torsion of solid shafts of noncircular cross section.

The **Prandtl stress function** method employs assumptions regarding stresses produced in the shaft by a torque, while the **Saint-Venant warping function** method is based on assumptions of the displacement field.

These two methods lead to the same solution. Here we begin with Saint-Venant's displacement assumptions but derive the governing equations in terms of the Prandtl stress function.



Governing Equations

Consider a straight shaft of constant cross section subjected to equal and opposite torques T at the ends, as shown in Figure. The origin of the coordinate system is taken to be at the **center of twist** (COT) of the cross section, about which the cross-section rotates during twisting under the torque.

Note that the cross-section here is not restricted to circular or hollow circular sections only. By the definition of the COT, the in-plane displacements vanish at this location.

For a circular cross section, the COT is obviously located at the center of the cross-section.
In general, the location of the COT depends on the shape of the cross section and how the end is supported.

For convenience, we assume that the Cartesian coordinate system is set up with the origin located at the C(?)

Let α denote the total angle of rotation (twist angle) at a section z relative to the end at $z = 0$. The rate of twist (twist angle per unit length) at z is denoted by

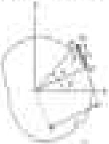
$$\theta = \alpha/z$$

Saint-Venant assumed that during torsional deformation, plane sections **warp**, but their projections on the x-y plane rotate as a **rigid body**. This assumption implies that the in-plane displacement components u and v follow those of a rigid body rotation.

Consider an arbitrary point P on the cross-section at z that moves through a rotation of a small angle α to P' after the torque is applied.

For clarity in illustration, we select P to locate on the lateral boundary of the bar (it could be any where inside).

For clarity in illustration, we select P to locate on the lateral boundary of the bar



$$u = -r\alpha \sin\beta = -\alpha y = -\theta z y \quad (4.1)$$

$$v = r\alpha \cos\beta = \alpha x = \theta z x \quad (4.2)$$

in which r is the distance from the origin of the coordinates to point P. This displacement field represents a rigid rotation of the cross-section through angle α in the x-y plane.

The displacement w in the z direction is assumed to be independent of z, and thus can be expressed as follows

$$w(x, y) = \theta\psi(x, y) \quad \text{[illegible]}$$

Where,
the rate of twist θ is independent of z and ψ(x, y) is the warping function.

The displacement field given by [illegible]–[illegible] yields

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \gamma_{xy} = 0$$

From the stress-strain relations, we conclude that

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

Thus, τ_{xz} and τ_{yz} are the only two nonvanishing stress components. In view of the displacement field, it is easy to see that τ_{xz} and τ_{yz} are independent of z.

In the absence of body forces, the equations of equilibrium are reduced to

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

Since the nonzero stresses are only shear stresses, the Prandtl stress function $\phi(x, y)$ will be the appropriate function in solving elasticity problems involving torsion.
It is shown that the relation between the Prandtl stress function $\phi(x, y)$ and shear stress components τ_{zx} and τ_{zy} are

$$\tau_{zx} = \frac{\partial \phi}{\partial y}, \qquad \tau_{zy} = -\frac{\partial \phi}{\partial x} \qquad (4.2)$$

From (4.1) and (4.2) and the strain-displacement relations :

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

we obtain

$$\gamma_{zx} = \frac{\partial w}{\partial x} - \theta y \qquad (4.6)$$

$$\gamma_{zy} = \frac{\partial w}{\partial y} + \theta x \qquad (4.7)$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$u = -r\alpha \sin\beta = -\alpha y = -\theta z y$$

$$v = r\alpha \cos\beta = \alpha x = \theta z x$$

Using (4.6) and (4.7), it is easy to derive the following equation:

$$\frac{\partial \gamma_{zy}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} = 2\theta \qquad (4.8)$$

This is the compatibility equation for torsion. Using the stress-strain relations

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

we obtain, from (4.8)

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = 2G\theta \qquad (4.9)$$

In terms of the Prandtl stress function, (4.9) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

Boundary Conditions

It can be observed from figure that there are basically three surfaces on a uniform bar – the lateral surface and the two plane sections normal to which torque is applied.

As such, on the lateral surface of the bar no loads are applied. Thus, the stress vector (traction) T must vanish. Using i.e.

$$\{t\} = [\sigma]\{n\}$$

The stress vector can be evaluated on the lateral surface by specifying the unit normal vector **n**. On the lateral surface, $n_z = 0$. Thus,

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & 0 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ 0 \end{Bmatrix}$$

Explicitly, we have

$$t_x = 0, \quad t_y = 0$$

$$t_z = \tau_{zx} n_x + \tau_{zy} n_y = \frac{\partial \phi}{\partial y} n_x - \frac{\partial \phi}{\partial x} n_y \quad \text{[illegible]}$$

Referring to Figure, it is easy to derive:

$$n_x = \sin \eta = \frac{dy}{ds} \quad \text{[illegible]}$$

$$n_y = \cos \eta = -\frac{dx}{ds} \quad \text{[illegible]}$$

Using the relations in (4.24) and (4.25), (4.18) can be expressed as

$$t_z = \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds}$$

The traction-free boundary condition $t_z = 0$ is now given by

$$\frac{d\phi}{ds} = 0 \quad \text{or} \quad \phi = \text{constant}$$

on the lateral surface. For solid sections with a single contour boundary, this constant is arbitrary and can be chosen to be zero. Thus, the boundary condition can be expressed as

$\phi = 0$ on the lateral surface of the bar

Torque-Stress Relations

It is obvious that the developed shear stresses τ_{xz} and τ_{yz} inside the bar are due to the applied torque T.

Consider a differential area $dA = dx\,dy$. The torque produced by the stresses in this area is:

$$dT = x\tau_{yz}\,dA - y\tau_{xz}\,dA$$
$$= \left[-x\frac{\partial\phi}{\partial x} - y\frac{\partial\phi}{\partial y}\right]dA$$

Shear stresses in the cross section

The total resultant torque is obtained by integrating dT over the entire cross section i.e.

$$T = -\iint_A \left[x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right] dx\,dy$$

$$= -\iint_A \left[\frac{\partial}{\partial x}(x\phi) - \phi \right] dx\,dy - \iint_A \left[\frac{\partial}{\partial y}(y\phi) - \phi \right] dx\,dy$$

$$= 2\iint_A \phi\,dx\,dy - \int \left[x\phi\right]_{x_1}^{x_2} dy - \int \left[y\phi\right]_{y_1}^{y_2} dx$$

where x_1, x_2, y_1, and y_2 are integration limits on the boundary. Since ϕ vanishes on the boundary contour, the last two terms of the equation above vanish. Thus,

$$T = 2\iint_A \phi\,dx\,dy \qquad (3.15)$$

The derivation above clearly indicates that the solution of the torsion problem lies in finding the stress function $\phi(x, y)$ that vanishes along the lateral boundary of the bar.
Once $\phi(x, y)$ is determined, the location of the COT $(x = 0, y = 0)$ is also defined.

Warping Displacement

For bars of arbitrary cross section, warping (out of plane displacement) of the cross section occurs when twisted. From (4.6 and 4.7) we have:

$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \theta y \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} - \theta x$$

The warping displacement w can be obtained by integrating the equations above:

$$w(x,y) = \frac{1}{G}\int \tau_{zx}\,dx + \theta \int y\,dx + f(y)$$

$$w(x,y) = \frac{1}{G}\int \tau_{zy}\,dy - \theta \int x\,dy + g(x)$$

where f(y) and g(x) are arbitrary functions [illegible]

Torsion Constant

The torsion equation $T = GJ\theta$ can be used to express the relation between the applied torque T and the resulting rate of twist θ for shafts of arbitrary cross sections. The torsion constant J is obtained as

$$J = \frac{T}{G\theta}$$

[illegible]

Using (4.[illegible]) and (4.[illegible]) in the equation above, we obtain

$$J = \frac{2\int \phi\,dx\,dy}{-\frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)} = -\frac{4}{\nabla^2 \phi}\int_A \phi\,dx\,dy$$

Thus, once the Prandtl stress function is solved, the **torsional rigidity** GJ of the shaft is also determined.

BARS WITH CIRCULAR CROSS SECTIONS

To find the suitable Prandtl stress function, consider a uniform bar of circular cross section. If the origin of the coordinates is chosen to coincide with the center of the cross section, the boundary surface is given by the equation:

$$x^2 + y^2 = a^2$$

[illegible]

where a is the radius of the circular boundary. Assume the stress function as

$$\phi = C\left(\frac{x^2}{a^2} + \frac{y^2}{a^2} - 1\right) \qquad (4.27)$$

For a circular bar any point on the lateral surface satisfies $x^2 + y^2 = a^2$.

On this surface:

$$\phi = C\left(\frac{a^2}{a^2} - 1\right) = 0$$

As such, the stress function (4.27) satisfies the boundary condition:

$$\phi = 0 \quad \text{on the lateral surface of the bar}$$

Substituting (4.27) into the compatibility (4.18), we have:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \qquad \text{[illegible]}$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = C\left(\frac{2}{a^2}+\frac{2}{a^2}\right) = \frac{4C}{a^2} = -2G\theta$$

Rearranging:

$$C = -\frac{1}{2}a^2G\theta$$

Substituting the expression of C in (4.27), the complete stress function that solves the torsion problem for a bar with circular cross section can be obtained:

$$\phi = -\frac{1}{2}a^2G\theta\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}-1\right) = -\frac{G\theta}{2}\left(x^2+y^2-a^2\right)$$

Thus, the above stress function solves the torsion problem. It also indicates that the center of the circular section is the COT.

Using the Torque eqn, we have the torque as:

$$T = 2\iint_A \phi\, dx\, dy$$

$$T = 2C\iint_A\left(\frac{x^2}{a^2}+\frac{y^2}{a^2}-1\right)dx\,dy$$

$$= 2C\iint_A\left(\frac{r^2}{a^2}-1\right)dA$$

$$= 2C\left(\frac{J}{a^2}-A\right)$$

Where,

$$J = \iint_A r^2\, dA = \frac{1}{2}\pi a^4$$

is the polar moment of inertia of the cross section, and

$$A = \pi a^2$$

is the cross sectional area [illegible]

Hence,

$a^2 A = 2I_p$ Hence,

$$T = -\frac{2CI_p}{a^2} = \theta G I_p$$

Put the C value in the above eqn. $C = -\frac{1}{2}a^2 G\theta$

$$T = \theta G I_p = \theta G J$$

where GJ is called the **torsional rigidity**

$2I_p$

$T = 2C\left(\frac{I_p}{a^2} - A\right)$

[illegible]

Using the Prandtl's equation the shear stresses are

$$\tau_{zx} = \frac{\partial\phi}{\partial y} = 2C\frac{y}{a^2} = -G\theta y \quad (4.27)$$

$$\tau_{zy} = -\frac{\partial\phi}{\partial x} = -2C\frac{x}{a^2} = G\theta x \quad (4.28)$$

$\phi = C\left(\frac{x^2}{a^2} + \frac{y^2}{a^2} - 1\right)$

$C = -\frac{1}{2}a^2 G\theta$

Consider a cylinder of a circular cross-section of radius r cut from the circular bar of radius a. On the lateral surface of this cylinder of radius r (see Figure 4.5a), the stress vector is given by [illegible]. Thus,

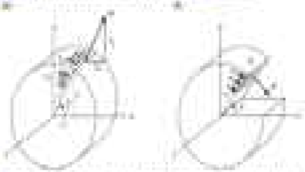


Fig. 4.5 Stress vectors on (a) a cylinder of radius r (r ≤ a) cut out of the original cylinder and (b) a surface cut along the radial direction of the cylinder

the stress vector is

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ 0 \end{Bmatrix}$$

$$t_x = t_y = 0$$

$$t_z = \tau_{xz} n_x + \tau_{yz} n_y$$

Also note that:

$$n_x = \cos\beta = \frac{x}{r}$$

$$n_y = \sin\beta = \frac{y}{r}$$

Using [illegible] and the relations above, we obtain:

$$t_z = -G\theta\frac{xy}{r} + G\theta\frac{xy}{r} = 0$$

As shown in Figure 4.5a, the radial shear stress τ_{rz} on the cross-section vanishes since it is equal to t_z.

Next consider the surface exposed by cutting along the radial direction of the cylinder as shown in Figure 4.5(c). The unit normal vector to the surface is given by

$$n_1 = 0, \quad n_2 = \sin\beta = \frac{x_3}{r}, \quad n_3 = -\cos\beta = -\frac{x_2}{r} \qquad (4.29)$$

Substituting (4.27) and (4.28) together with (4.29) into (4.33) yields the only nonvanishing component of the stress vector in the x_1-direction as

$$t_1 = -G\theta r$$

$$\boxed{\begin{aligned} \sigma_{12} &= -G\theta x_3 \\ \sigma_{13} &= G\theta x_2 \end{aligned}}$$

$$\boxed{t_1 = \sigma_{12} n_2 + \sigma_{13} n_3}$$

On the x_1-face (the cross-section), the tangential shear stress τ (that is perpendicular to the radial direction) is equal to t_1 in magnitude. Adjusting the sign for direction, we have

$$\tau = -t_1 = G\theta r$$

Using (4.30) the magnitude of the resultant shear stress can be expressed in terms of the torque

$$\tau = \frac{Tr}{J}$$

$$\boxed{T = \theta G J} \quad (4.30)$$

It is evident that the magnitude of τ is proportional to r. This is the well known result for torsion of circular bars.

Using the following equations we can prove:

$$\tau_x = \frac{\partial w}{\partial x} - \theta y \qquad \tau_y = \frac{\partial w}{\partial y} + \theta x \qquad \text{and} \qquad \tau_{xz} = -G\theta y \qquad \tau_{yz} = G\theta x$$

$$\psi = 0$$

Thus, we have with circular cross-sections under torsion, there is no warping.

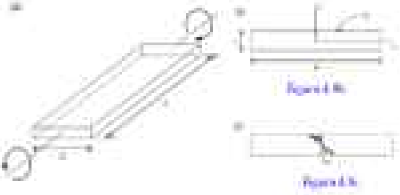
BARS WITH NARROW RECTANGULAR CROSS SECTIONS

The result for circular cross-sections cannot be extended automatically to noncircular cross-sections. For example, for a square cross-section, the shear stress cannot be assumed to be perpendicular to the radial direction and its magnitude is not proportional to the radial distance.

Further, warping is present.

Consider a bar (shaft) with a narrow rectangular cross-section subjected to a pure torque as shown in Figure.

From the consideration of symmetry, the SST for this cross-section is located at the geometric center of the section.



Bar with a narrow rectangular cross section

To satisfy Saint-Venant's principle, the length L of the bar is assumed to be much greater than the width b of the cross-section. Moreover, it is assumed that the cross-section [see Figure 4.9(b)] of the bar is such that $b \gg t$.

Basically this is a thin plate along in the x-direction. For the top and bottom faces ($y = \pm t/2$), the traction-free boundary condition requires that

$$\tau_{yz} = 0$$

In terms of the stress function, this says that

$$\frac{\partial \phi}{\partial x} = -\tau_{yz} = 0 \quad \text{on the top and bottom faces}$$

Since t is very small, and τ_{zx} must vanish at $y = \pm t/2$, it is unlikely that the shear stress τ_{zx} would build up across the thickness.

Therefore, we can assume that $\tau_{zx} \approx 0$ throughout the thickness.

Consequently, we assume that ϕ is independent of x.

$$\frac{d^2\phi}{dy^2} = -2G\theta$$

Integrating the above equation twice, we obtain

$$\phi = -G\theta y^2 + C_1 y + C_2$$

The boundary condition requires that

$$\phi = 0 \quad \text{at} \quad y = \pm\frac{t}{2}$$

which leads to $\quad C_1 = 0 \quad$ and $\quad C_2 = G\theta\frac{t^2}{4}$

and subsequently

$$\phi = -G\theta\left[y^2 - \frac{t^2}{4}\right] \qquad (4.42)$$

The corresponding shear stresses are obtained from (4.5) and (4.42) as

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} \quad (4.5)$$

$$\tau_{xz} = -2G\theta y, \qquad \tau_{yz} = 0 \quad (4.43)$$

The shear stress τ_{xz} acts parallel to the x-axis and is distributed linearly across the width, as shown in figure 4.9C. The maximum shear stress occurs at $y = \pm t/2$, i.e.

$$(\tau_{xz})_{max} = G\theta t$$

The torque is obtained by substituting (4.42) into (4.19)

$$T = -2G\theta \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} \left(y^2 - \frac{t^2}{4} \right) dy\, dx$$

$$= \frac{bt^3}{3} G\theta$$

$$T = 2 \iint \phi\, dx\, dy \quad (4.19)$$

Define the torsion constant J as $J = \frac{bt^3}{3}$ (4.45)

Then $T = GJ\theta$

where GJ is the torsional rigidity.

From [illegible] and [illegible], we have:

$$\gamma_{xz} = \frac{\partial w}{\partial x} - \theta y \quad [illegible]$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \theta x \quad (4.7)$$

$$[illegible] \quad [illegible]$$

The amount of warping on the cross-section can be obtained from integrating the expressions above. We obtain:

$$w = -xy\theta$$

The integration constant is set equal to zero because $w = 0$ at the [illegible]. In fact, $w = 0$ along the centerline of the sheet.

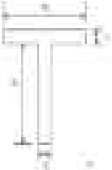
The results obtained here can be used for sections composed of a number of thin-walled members. For example, the T-section shown in Figure can be considered as a section consisting of two rectangular sections. The combined torsional rigidity is given by

$$GJ = G(J_1 + J_2)$$

where

$$J_1 = \frac{1}{3} b_1 t_1^3$$

$$J_2 = \frac{1}{3} b_2 t_2^3$$



The formula given by (4.42) can also be applied to curved open thin-walled sections by interpreting b as the total arc length as depicted by below Figure.

It is noted that the torsion constant J given by (4.43) is valid only if b/t is large. If the two dimensions are comparable then J should be evaluated using the elasticity solution obtained by solving a rectangular cross section using the Prandtl stress function method (Timoshenko and Goodier 1970). A correction coefficient β needs to modify the torsion constant of (4.45), i.e.

$$J = \beta \frac{bt^3}{3}$$

b/t	β
1.0	0.422
1.5	0.588
2.0	0.687
5.0	0.873
10.0	0.936
∞	1.0

For b/t = 1.0, 1.5, 2.0, 5.0, 10.0, β = 0.422, 0.588, 0.687, 0.873, 0.936, respectively

CLOSED SINGLE-CELL THIN WALLED SECTIONS

Members with closed thin-walled sections are quite common in aircraft structures.

Figure shows a closed thin-walled section with a single cell.

The wall thickness t is assumed to be small compared with the total length of the complete wall arc.

Wall section enclosed by an inner contour C0 and an outer contour C1.

It can be noticed that shear stress along the thickness direction is negligible in a thin rectangular section.

For such geometry, solving problems in the Cartesian coordinate system is convenient by setting the axes along the thickness and width directions.

Working with Cartesian system is not meaningful when the geometry of the thin section is irregular. It will be shown that setting a new coordinate system (s-n coordinates) where one axis lies along the width and the other axis lies along the thickness will simplify the analysis.

The wall section is enclosed by the inner contour S_1 and the outer contour S_0 as shown in Figure.

Using the Prandtl stress function ϕ, the stress-free boundary conditions are given by [ ]

$$\frac{d\phi}{ds} = 0 \quad \text{on } S_1 \text{ and } S_0$$

Thus, $\phi = C_0$ on S_0

$\phi = C_1$ on S_1

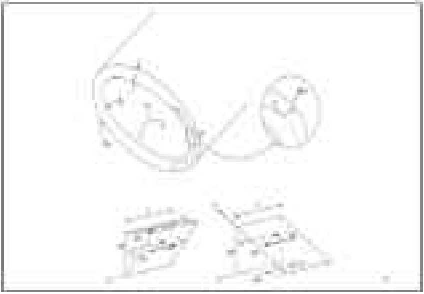
where C_0 and C_1 are two different constants and cannot be set equal to zero simultaneously as in the case of solid sections with a single boundary contour.

Consider the shear stresses at an arbitrary point on the wall section.

Let us set up a coordinate system sn so that s coincides with the centerline of the wall and n is perpendicular to s as shown in Figure.

Take an infinitesimal prismatic element of unit length in the z-direction as shown in Figure 6.

The active shear stresses on the side faces are shown in the figure. Note that the outward normal is perpendicular to the s-direction.



The equilibrium condition (balance of forces in the s-direction) gives:

$$\tau_s\, ds = -\tau_{zx}\, dy + \tau_{zy}\, dx$$

$$\tau_s = -\tau_{zx}\frac{dy}{ds} + \tau_{zy}\frac{dx}{ds}$$

$$= -\frac{\partial\phi}{\partial y}\frac{dy}{ds} - \frac{\partial\phi}{\partial x}\frac{dx}{ds}$$

$$= -\frac{d\phi}{ds}$$

Note that a negative sign is added in front of $\tau_{zx}\, dy$ to account for the fact that an increment ds is accompanied by a decrement dy.

Similarly, using the free body of figure c and the equilibrium condition, we have:

$$\tau_{zt}\, dt = \tau_{zx}\, dy - \tau_{zy}\, dx$$

$$\tau_{zt} = \tau_{zx}\frac{\partial y}{\partial t} - \tau_{zy}\frac{\partial x}{\partial t}$$

$$= \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t}$$

$$= \frac{\partial \phi}{\partial t}$$

Since $\tau_{zt} = \partial\phi/\partial t = 0$ on S_0 and S_1, and t is small, the variation of τ_{zt} across the wall thickness is negligible.

Hence, a reasonable approximation is to assume $\tau_{zt} = 0$ over the entire wall section.

As a result of this assumption, the τ_{zs} is retained as the only outstanding stress component.

Prandtl Stress Function

Let ϕ be expressed in terms of the coordinates s and n and expand ϕ in series of n as

$$\phi(s,n) = \phi_0(s) + n\phi_1(s) + n^2\phi_2(s) + \cdots$$

Where: $-\frac{t}{2} \le n \le \frac{t}{2}$

in which $t(s)$ is wall thickness and, in general, is a function of the wall contour.

Since the range of n is small, the high order terms of n in the eqn. can be neglected without causing much error. Retaining the linear term in the eqn., we have

$$\phi(s,n) = \phi_0(s) + n\phi_1(s) \qquad \text{Eq. 1}$$

The boundary conditions require that

$$\phi\left(s, \frac{t}{2}\right) = \phi_0 + \frac{t}{2}\phi_1 = C_0 \quad \text{on } S_0$$

$$\phi\left(s, -\frac{t}{2}\right) = \phi_0 - \frac{t}{2}\phi_1 = C_1 \quad \text{on } S_1$$

Solving the two equations, we obtain

$$\phi_0 = \frac{1}{2}(C_0 + C_1)$$

$$\phi_1 = \frac{1}{t}(C_0 - C_1)$$

Shear Flow q

The shear stress τ on the wall section in the s-direction is given by

$$\tau = \tau_s = -\frac{\partial \phi}{\partial n} = -\phi_{,n} = \frac{1}{t}(C_1 - C_2)$$

Thus, the shear stress τ over the thin-walled section is uniform over the thickness. Nevertheless, τ is still a function of the contour s if the wall thickness t is not constant.

Define the **shear flow** q (force / contour length) as

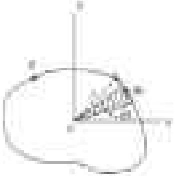
$$q = \tau t = C_1 - C_2$$

This indicating that regardless of the wall thickness, the shear flow is constant along the wall section.

Shear Flow-Torque Relation

The shear stress τ along the wall is usually represented by the shear flow q along the centerline of the wall. Since the shear flow forms a closed contour, the force resultants are equal to zero, i.e. $\Sigma F_x = 0$ and $\Sigma F_y = 0$. However, the shear flow produces a resultant torque.

Consider a constant shear flow q on a closed thin-walled section as shown in Figure. The resultant torque produced by the shear flow acting on the element segment ds is given by



$$dT = \rho q\, ds$$

Where ρ is the distance from the origin of the coordinates to the line segment ds.

The torque T can be obtained by integrating the above along the entire shear flow contour.

Noting that $\rho\, ds = 2dA$, we rewrite the integral above as an area integral:

$$T = \iint_A 2q\,dA = 2q\bar{A}$$

where $\bar{A}$ is the area enclosed by the shear flow or, equivalently, the area enclosed by the center line of the wall section.

Consider a shear flow q as shown in Figure. It can easily be shown that the resultant force R is oriented parallel to the line connecting the two end points P and Q of the shear flow and the magnitude of the resultant force is given by

$$R = qd$$

The components of the resultant force are

$$F_x = qh$$

$$F_y = qb$$

The torque about the z axis is

$$T = 2\bar{A}q$$

where $\bar{A}$ is the area bounded by the contour q and lines $\overline{OP}$ and $\overline{OQ}$.

The actual location e (see Fig) of the resultant force can be obtained from the torque equivalence condition, i.e.:

$$Re = T = 2\bar{A}q$$

Twist Angle

The torsion formula $T = GJ\theta$ can be considered as the universal equation that is valid for torsion on all geometries.

For a thin rectangular section or circular shaft or any geometries of which torsional constant J is readily available or known, the twist angle can be obtained by rearranging the torsion formula:

$$\theta = \frac{T}{GJ}$$

When torsion constant J is not known, then the above eqn. cannot be used to find the twist angle. In such a scenario, the twist angle is obtained first. The torsion constant J is then obtained using the torsion formula.

Method 2

Using the shear strains given by [4.5] and [4.7] and the stress-strain relations, we obtain:

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - \theta y\right) \qquad [4.10]$$
$$\tau_{yz} = G\left(\frac{\partial w}{\partial y} + \theta x\right)$$

Using the first line of [4.10], we have:

$$T = \tau_{zs} = -\tau_{xz}\frac{dy}{ds} + \tau_{yz}\frac{dx}{ds} \qquad [4.11]$$

Integrating τ over the closed contour along the centerline of the wall, we have:

$$\oint \tau\, ds = G \oint \frac{\partial u}{\partial s} ds + G\theta \oint \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right) ds$$

$$= G u\Big|_0^L + G\theta \oint (x\, dy - y\, dx)$$

where L is the total length of the contour. The first term on the right-hand side of the above eqn. vanishes because $u(0) = u(L)$. The second term can be integrated using Green's theorem, which states that

$$\oint (f\, dx + g\, dy) = \iint_A \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

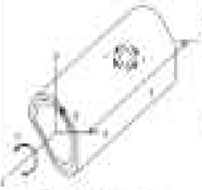
By identifying g as x and f as −y in the above eqn., one can use Green's theorem to obtain:

$$\oint \tau\, ds = 2G\theta \bar{A}$$

from which θ is obtained as:

$$\theta = \frac{1}{2G\bar{A}} \oint \tau\, ds = \frac{1}{2G\bar{A}} \oint \frac{q}{t} ds \qquad \text{[illegible]}$$

Method 2 for Constant Shear Flow



Thin-walled bar of unit length.

Consider a thin-walled bar of unit length as shown in figure. The shear stress is

$$\tau = \frac{q}{t}$$

and the shear strain is

$$\gamma = \frac{\tau}{G} = \frac{q}{Gt}$$

The corresponding strain energy density is given by

$$W = \frac{1}{2}\tau\gamma = \frac{q^2}{2Gt^2}$$

The total strain energy stored in the bar (of unit length) is

$$U = \int W \, dA$$

$$= \oint \frac{q^2}{2Gt} \, ds$$

The work done by torque T through the twist angle θ is given by

$$W_e = \frac{1}{2}T\theta = \frac{1}{2}\cdot 2q\bar{A}\theta = q\bar{A}\theta$$

In the equation just given, $\bar{A}$ is the area enclosed by the shear flow contour. From the energy principle (work done by external forces is equal to the total strain energy) we obtain

$$W_e = U$$

or explicitly

$$q\bar{A}\theta = \frac{q^2}{2G}\oint\frac{ds}{t}$$

Thus,

$$\theta = \frac{q}{2\bar{A}G}\oint\frac{ds}{t}$$

This is identical to (3.42) if q is constant along the wall.

Torsion Constant J

Since $T = 2q\bar{A}$ we have

$$\frac{T}{GJ} = \frac{q}{2\bar{A}G}\oint\frac{ds}{t}$$

From this relation we obtain the torsion constant J for the single-cell thin-walled section as

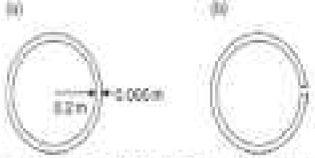
$$J = \frac{2AT}{q\oint \frac{ds}{t}}$$

$$= \frac{4A^2}{\oint \frac{ds}{t}}$$

in which $T = 2qA$ has been used.

EXAMPLE

Compare the torsional rigidities of the two tubes shown in Figure



Thin-walled tubes with (a) a closed section and (b) a slit section

Closed Hollow Section

As shown in Figure, the wall thickness is $t = 0.005$ m and the average radius is 0.2025 m. Thus,

$$\bar{A} = \pi(0.2025)^2 = 0.129\ m^2$$

$$\oint \frac{ds}{t} = \frac{\pi \times 0.405}{0.005} = 254$$

From (3.60), the torsion constant is obtained as

$$J_1 = \frac{4\bar{A}^2}{\oint \frac{ds}{t}} = \frac{4 \times (0.129)^2}{254} = 2.62 \times 10^{-4}\ m^4$$

Slit Section

The slit section is made by cutting open the closed section (Figure), then the torsion constant is given by (4.41) as

$$J_2 = \frac{bt^3}{3} = \frac{\pi \times 0.4 \times (0.005)^3}{3} = 5.24 \times 10^{-8}\ m^4$$

ratio of torsional rigidities of these two tubes is

$$\frac{J_1}{J_2} = 5000$$

It is evident that the tube with the closed section has a much higher torsional rigidity than the slit tube.

EXAMPLE

Find the maximum torque and allowable shear flow for the three-stringer thin-walled beam with the cross section as shown in Figure if the allowable shear stress of the material is 200 MPa.

Three-stringer thin-walled bar

Solution

The contribution of individual stringers to the overall torsional rigidity of the thin-walled structure is small and can be neglected. Hence, this structure can be considered as a single-cell closed section with a nonuniform wall thickness and the shear flow is constant along the wall.

If the torque T (N·m) is given, then the shear flow is obtained from the relation $T = 2\bar{A}q$. The area $\bar{A}$ is readily obtained as

$$\bar{A} = \frac{1}{2}\pi(0.6)^2 + \frac{1}{2}(2 \cdot 1.2) = 1.765 \text{ m}^2$$

thus,

$$q = \frac{T}{2\bar{A}} = \frac{T}{3.53} \text{ N/m}$$

This twist angle is obtained using (3.46). We obtain

$$\theta = \frac{1}{2\bar{A}G} q \oint \frac{ds}{t}$$

$$= \frac{q}{2 \times 1.765G} \left(\frac{1.2\pi}{2t_1} + \frac{2}{t_2} + \frac{2.33}{t_3} \right)$$

$$= 78 \frac{T}{G} \text{ rad/m}$$

Because of its smaller thickness, the shear stress in the curved wall is higher than that in the straight walls. The shear stress in the curved wall is

$$\tau = \frac{q}{t_1} = \frac{T}{0.005 \times 3.53} = 56.66T$$

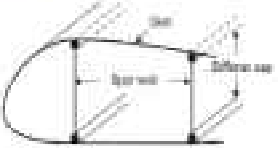
If the allowable shear stress of the material is 200 MPa, then the maximum torque that the structure can take is

$$T_{max} = \frac{\tau_{allow}}{56.66} = \frac{200 \times 10^6}{56.66} = 3.53 \times 10^6 \text{ N m}$$

MULTICELL THIN WALLED SECTIONS

Wing sections are often composed of airfoil skins supported by thin vertical walls to form multicell constructions.

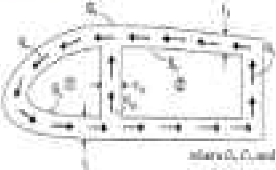
Figure shows a two-cell skin-web section.



Two-cell stringer-skin-web section

In addition, stiffeners are used to carry bending loads. The individual stiffeners, although having large concentrated cross-sectional areas, have relatively small torsion constants and do not make a significant contribution to the torsional rigidity of the wing box and are often neglected in the consideration of torsional stiffness of the wing box.

For torsion of a single-cell section, the Prandtl stress function must be constant along each boundary contour. For the two-cell section, there are three boundary contours, i.e. S_0, S_1, and S_2 (see Figure). Thus, we have



$$\phi(S_0) = C_0$$

$$\phi(S_1) = C_1$$

$$\phi(S_2) = C_2$$

where C_0, C_1 and C_2 are three different constants.

From the result of (4.22), we note that the shear flow between two bounding contours is equal to the difference between the values of ϕ along these contours. Specifically, for each cell, the shear flow is considered positive if it forms a counterclockwise torque about the cell and its value is equal to the value of ϕ on the inside contour minus that on the outside contour, i.e.

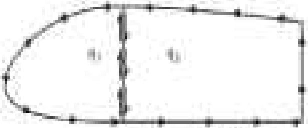
$$
\begin{aligned}
q_1 &= C_1 - C_0 \\
q_2 &= C_2 - C_0 \qquad (4.71) \\
q_{12} &= C_1 - C_2
\end{aligned}
$$

Note that the direction of q_{12} assumed in figure is positive for cell 1 and negative for cell 2.

From (4.71), we obtain

$$q_{12} = q_1 - q_2$$

In view of this relation, the shear flow system can be viewed as the superposition of two shear flows q_1 and q_2 as depicted in the figure.



Superposition of two constant shear flows

Thus the torque contributed by each cell can be calculated by using

$$T = 2\bar{A}q$$

The total torque of the two-cell section is:

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 \qquad (4.73)$$

where $\bar{A}_1$ and $\bar{A}_2$ are the areas enclosed by the shear flows q_1 and q_2 respectively.

The twist angles θ_1 and θ_2 of the cells are obtained using (4.65):

$$\theta_1 = \frac{1}{2\bar{A}_1 G} \oint_{cell\,1} \frac{q\,ds}{t}$$

$$\theta_2 = \frac{1}{2\bar{A}_2 G} \oint_{cell\,2} \frac{q\,ds}{t}$$

It is important to note that for each cell, the shear flow (and twist angle) is taken positive if it flows (rotates) in the counterclockwise direction. For the shear flow in Figure, $q_{12} = q_1 - q_2$ should be used for cell 1, while for cell 2, $-q_{12} = q_2 - q_1$ should be used.

Since the entire thin-wall section must rotate as a rigid body in the plane, we require the compatibility condition

$$\theta_1 = \theta_2 = \theta \qquad (4.75)$$

Equations (4.73) and (4.75) are solved to find the two unknown shear flows q_1 and q_2.

Sections with more than two cells can be treated in a similar way. Additional equations provided by the compatibility condition are available for solving additional unknown shear flows. The torque for n-cell section is given by

$$T = \sum_{i=1}^{n} 2A_i q_i$$

The twist angle θ of the section is the same as the individual cells. Hence, we choose cell 1 to calculate the twist angle.

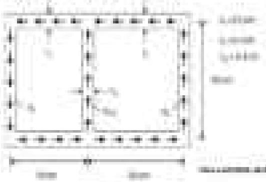
$$\theta = \theta_1 = \frac{1}{2A_1 G} \oint_{cell\ 1} \frac{q\,ds}{t}$$

From $T = GJ\theta$, we obtain the torsion constant J of the multicell section as

$$J = \frac{T}{G\theta} = \frac{4\bar{A}_i \sum \bar{A}_j q_j}{\oint_{cell\ i} q\, ds/t}$$

EXAMPLE

A two-cell thin-walled box beam (see figure) is subjected to a torque T that causes a twist angle $\theta = 1°/m$ (0.0175 rad/m). Assume that $G = 27$ GPa. Find the corresponding shear flow and torque.



$$\theta = \frac{1}{2GA}\oint \tau\, ds = \frac{1}{2GA}\oint \frac{q}{t}\, ds \qquad (4.67)$$

$$\theta_1 = \frac{1}{2GA_1}\left[\frac{q_1(0.4 + 0.3 + 0.4)}{t_1} + \frac{q_{12}(0.3)}{t_{12}}\right]$$

Substituting measured values into the equation above, we have

$$0.087 = 7.56 \times 10^{-6} q_1 - 1.55 \times 10^{-6} q_2$$

Similarly for cell 2,

$$\theta_2 = \frac{1}{2GA_2}\left[\frac{q_2(1.5)}{t_2} - \frac{(q_1 - q_2)(0.3)}{t_{12}}\right]$$

$$0.087 = -1.24 \times 10^{-6} q_1 + 4.01 \times 10^{-6} q_2$$

Solving (4.71) and (4.72), we obtain the shear flows as

$$q_1 = 1.7 \times 10^{4}\ \text{N/m}$$

$$q_2 = 2.7 \times 10^{4}\ \text{N/m}$$

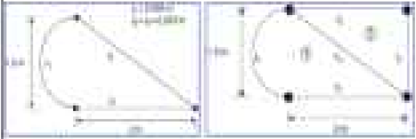
The torque that produces the given twist angle is

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 = 3.03 \times 10^{4}\ \text{N}\cdot\text{m}$$

The torsion constant is

$$J = \frac{T}{G\theta} = \frac{2.03 \times 10^{[illegible]}}{(27 \times 10^{9}) \times 0.007} = 1.08 \times 10^{-[illegible]}\ m^4$$

Compare the torsional rigidities of these two beams.



$$T = \frac{1}{[illegible]}G\theta = [illegible]\ kN \cdot m$$

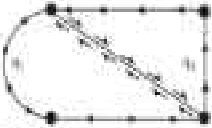
$$J_1 = \frac{\pi d^{[illegible]}}{[illegible]} \quad J = 0.014\ m^4$$

For the single-cell beam, the torsional rigidity can be obtained as

$$T = \frac{1}{79} G\theta = 0.013 G\theta \ \text{N·m}$$

and the torsion constant $J = 0.013 \ \text{m}^4$.

$$J = \frac{4A^2}{\oint \frac{ds}{t}}$$



Shear flows in a two-cell section

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 \quad (4.73)$$

For the two-cell section of Figure 4.21 we have $\bar{A}_1 = 1.367 \ \text{m}^2$ and $\bar{A}_2 = 1.2 \ \text{m}^2$ for cells 1 and 2, respectively. From (4.73) we obtain

$$T = 2.73 q_1 + 2.4 q_2 \quad (4.80)$$

in which the shear flows q_1 and q_2 are indicated in Figure 4.25. The rate of twist for cell 1 is

$$\theta_1 = \frac{1}{2 \times 1.367 G}\left[\frac{1.2\pi}{2t_1} q_1 + \frac{2}{t_2} q_1 + \frac{2.25}{t_3}(q_1 - q_2)\right] \quad (4.81)$$

$$\theta = \frac{1}{2\bar{A}G}\oint \frac{q\,ds}{t}$$

Similarly, for cell 2 we have

$$\theta_2 = \frac{1}{2 \times 1.20}\left[\frac{2.33}{t_1}(q_2 - q_1) + \frac{1.2}{t_2}q_2 + \frac{2}{t_2}q_2\right]$$

Finally, the compatibility eq. (4.71) leads to

$$\frac{1}{1.968}\left[\frac{1.2\pi}{2t_1}q_1 + \frac{2}{t_1}q_1 + \frac{2.33}{t_1}(q_1 - q_2)\right]$$

$$= \frac{1}{1.2}\left[\frac{2.33}{t_1}(q_2 - q_1) + \frac{1.2}{t_2}q_2 + \frac{2}{t_2}q_2\right] \qquad (4.82)$$

$$841.5\, q_1 - 904.11\, q_2 = 0$$

Solving (4.80) and (4.82) simultaneously for q_1 and q_2, we obtain:

$$q_1 = 0.177T, \quad q_2 = 0.165T$$

Since the two cells have identical twist angle, we can use (4.81) to calculate the rate of twist, with the result

$$\theta = \theta_1 = \frac{32.9}{G}T \quad \text{or} \quad T = 0.0304G\theta$$

Thus, the two cell section has a torsion constant $J = 0.03\ m^4$, which is more than twice that of the single cell section.

Consider the single-cell section obtained from that of Figure [illegible] by removing the diagonal sheet. The shear flow is easily obtained from

$$T = 5.93q$$

Subsequently, the rate of twist is obtained as

$$\theta = \frac{13.8}{G}T$$

and the torsion constant is $J = 0.0296\ m^4$

The value is almost equal to that of the two-cell section. From this example, it is interesting to note that torsional rigidity of a closed thin-walled section may not be increased significantly by compartmentalization.

WARPING IN OPEN THIN WALLED SECTIONS

Except for circular cross sections, shafts of noncircular sections exhibit warping under pure torques.
In other words, out-of-plane displacements occur during torsion.

For instance, in the narrow rectangular section, the out-of-plane displacement is given by

$$w = -xy\theta \qquad ([illegible])$$

Substituting ([illegible]) in the following eq., one has

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \theta y \qquad \gamma_{zy} = 0$$

which is consistent with initial assumption $\gamma_{sz} = 0$. Moreover, we note that $\gamma_{sz} = 0$ along the centerline of the wall from (4.42). Thus, we have

$$\gamma_{sz} = 0 \qquad \text{along } n = 0.$$

It is noted that $w = 0$ along the centerline ($n = 0$) of the wall. Thus, warping occurs only across the thickness of the wall. This type of warping is usually called **secondary warping**.

For general thin-walled sections, the centerline of the wall may also warp with the magnitude much greater than the secondary warping. This is known as **primary warping**.

Consider a curved thin-walled section of uniform thickness as shown in Figure. Following the procedure described in previous Section, we set up a right-hand coordinate system s-n-z so that s coincides with the centerline of the wall, n is perpendicular to s, and z remains unchanged.

The origin of s can be chosen arbitrarily.

At a point along the s contour, let w denote the displacement in the z-direction and u_s the displacement in the s-direction. With respect to this new coordinate system, the shear strains are

$$\gamma_{sz} = \frac{\partial w}{\partial s} + \frac{\partial u_s}{\partial z}, \quad \gamma_{nz} = \frac{\partial w}{\partial n} + \frac{\partial u_n}{\partial z} \quad (4.46)$$

In view of $\gamma_{sz} = 0$, we have $\gamma_{sz} = 0$ along s (the centerline of the wall). Thus,

$$\frac{\partial w}{\partial s} = -\frac{\partial u_s}{\partial z} \quad (4.47)$$

Apparently, the out-of-plane displacement along s may be obtained by integration of when $\partial u_s/\partial z$ is known.

For convenience, we assume that the CST for the thin-walled section is known. Figure shows the relative position of the CST and the contour s. Take an arbitrary point P on the contour s. After the application of a torque, point P moves to P' because of the rigid rotation assumption of Saint Venant's torsion formulation.

fig 4.11

The total displacement is $\overline{PP'} = r\alpha = rz\theta$

where α is the total twist angle measured from $z = 0$ to the current section of interest.

From Figure 4.27a, the displacement u_s in the s-direction (i.e. the tangential direction at point P) is

$$u_s = \overline{PP'} \cos\beta = rz\theta \cos\beta = \rho z\theta \qquad (4.48)$$

where ρ is the distance from the CST to the tangent line at point P as shown in Figure 4.27a.

Substituting (4.48) in (4.37), we obtain

$$\frac{\partial w}{\partial s} = -\rho\theta \qquad (4.49)$$

To perform the integration of (4.49) along s, we consider a line segment ds along the contour s as shown in Figure 4.27a. The area $d\bar{A}$ is recognized to be

$$d\bar{A} = \frac{1}{2}\rho\, ds \qquad (4.50)$$

Now, integrating (4.20) with the aid of (4.30) we obtain

$$w(s) - w(0) = -\theta \int_0^s \rho \, ds = -\theta \oint_{A_s} 2\, d\hat{A} = -2\hat{A}_s \theta \qquad (4.31)$$

Where $d\hat{A}$ is the area enclosed by the contour s and the two lines connecting the CDT with the two points $s = 0$ and $s = s$ (see Figure 4.15c). The area can also be considered as the area swept by the generator line (the line connecting CDT and the origin of ρ) from $s = 0$ to $s = s$. If the origin of s is selected such that warping vanishes at $s = 0$ then $w(0) = 0$.

EXAMPLE

The CDT of the thin-walled channel section shown in Figure is located on the horizontal axis of symmetry at a distance

$$e = \frac{tb^2 h^2}{I_x}$$

[illegible]

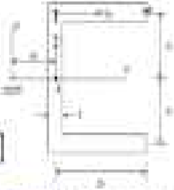


Fig. 4.16 Thin walled channel section.

to the left of the vertical web, where I_x is the moment of inertia about the x axis. Find the warping displacement.

Consider the contour s which is broken into two straight parts s_1 and s_2 as shown in Figure 4.29. For contour s_1 we have

$$\bar{A}_{s_1} = \frac{1}{2} e s_1$$

and for contour s_2

$$\bar{A}_{s_2} = -\frac{1}{2} h s_2$$

in which a negative sign is added because the direction of s_2 produces a clockwise rotation about the CoS.

From (4.91), the warp of the thin wall is

$$w(s_1) = w(0) - 2\bar{A}_{s_1}\theta = w(0) - e s_1 \theta \quad \text{on } s_1$$

Warping at location 2 (put $s_1 = h$) = $-eh\theta$ (s_1 = 0 to h)

and (s_2 = 0 to b)

$$w(s_2) = w(s_1 = h) - 2\bar{A}_{s_2}\theta = w(0) - eh\theta + h s_2 \theta \quad \text{on } s_2$$

Warping at location 3 (put $s_2 = b$) = $-eh\theta + hb\theta = \theta(hb - eh)$

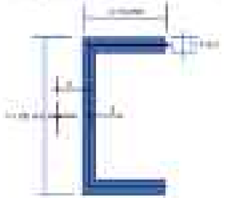
The warping displacement in the lower half of the section can easily be recognized to be of the same magnitude as the upper half except for the sign change resulting from the fact that the s contour is in the negative direction.

In view of the antisymmetric warping of the section, we conclude that $w(0) = 0$. In fact, this is true at locations on any axis of symmetry of the thin-walled section.

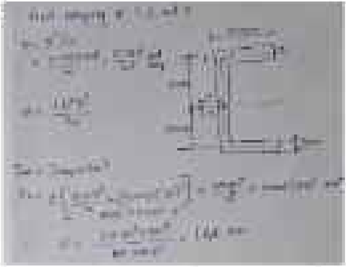
Thus, the warping displacement at the upper corner point is $-ehb$ (a negative sign means that the displacement is in the negative z-direction), and at the upper free edge it is $(bh - eh)\theta$.

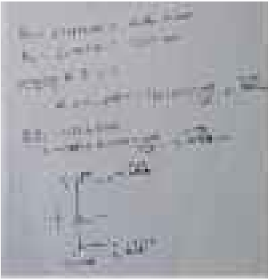
Tutorial

- Find the warping displacements at the location a,b and c, take $\theta = 1°/m$.



Tutorial





WARPING IN CLOSED THIN-WALLED SECTIONS

We follow the same procedure used in treating open sections. First, set up the s-n-z coordinate system with the origin at the CT and with positive s forming counterclockwise about the CT.

Shear flow in a closed thin-walled section.

The shear flow q is related to the shear stress τ_{sz} as

$$q_s = t\tau_{sz} = tG\gamma_{sz} \qquad (4.92)$$

where t is the thickness of the wall and G is the shear modulus. Using the first equation in (4.33), we obtain from (4.92) the following relation:

$$q_s = tG\left(\frac{\partial w}{\partial s} + \frac{\partial u_s}{\partial z}\right) \qquad (4.93)$$

Using (3.88), which is valid for closed sections, we obtain from (4.93) the equation

$$dA_s = \rho\, ds$$

$$\frac{\partial w}{\partial s} = \frac{q_s}{Gt} - \rho\theta' \qquad (4.94)$$

Integration of (4.94) along s leads to

$$w(s) - w(0) = \int_0^s \frac{q_s}{Gt}\, ds - \theta' \int_0^s \rho\, ds$$
$$= \int_0^s \frac{q_s}{Gt}\, ds - 2A_s\theta' \qquad (4.95)$$

This gives the warping at any point relative to that at the point s = 0.

EXAMPLE

The cross-section of a thin-walled box beam subjected to a torque T is shown in Figure 4.11. Find the warping displacement of the cross-section.

Fig. 4.11 Thin-walled box beam with four stringers

The axes of symmetry of this section are chosen to coincide with x and y axes, respectively, as shown in Figure 4.21. From symmetry, the CDT is seen to be at the origin of the coordinate system. Moreover, we take the origin for s at ($x = a/2$, $y = 0$).

From symmetry, we have $w(0) = 0$. In fact, any midpoint of the four sidewalls can be selected as the origin of s and satisfies $w(0) = 0$.

For closed single-celled sections, we have

$$q_s = \frac{T}{2\bar{A}} \quad (4.96) \quad \text{and} \quad \theta = \frac{1}{2G\bar{A}}\oint \frac{q}{t}\, ds = \frac{T}{4G\bar{A}^2}\oint \frac{ds}{t} \quad (4.97)$$

in which $\bar{A}$ is the total area enclosed by the thin walls

Substituting (4.96) and (4.97) in (4.95), we have the warping along the thin wall as

$$w(s) = \frac{T}{2\bar{A}G}\left(\int_0^s \frac{ds}{t} - \frac{\bar{A}_s}{\bar{A}}\oint \frac{ds}{t}\right)$$

$$= \frac{T}{2\bar{A}Gab}\left[\int_0^s \frac{ds}{t} - 2\bar{A}_s\left(\frac{1}{bt_1} + \frac{1}{at_2}\right)\right] \quad (4.98)$$

By dividing the contour s into five consecutive segments, 01, 12, 23, 34, and 45, the integral in eq (4.98) can easily be evaluated on each segment. For 01, we obtain

$$w(s_1) = \frac{T}{2Gab}\left[\frac{s_1}{t_1} - \frac{as_1}{2}\left(\frac{1}{bt_1} + \frac{1}{at_2}\right)\right]$$

which is linear in s_1.
At stringer 1, $s_1 = b/2$ and the warp is

$$w_1 = w\left(\frac{b}{2}\right) = \frac{T}{8Gab}\left(\frac{b}{t_2} - \frac{a}{t_1}\right)$$

(4.99)

The warp along s_2 is

$$w(s_2) = w_1 + \frac{T}{2Gab}\left[\frac{s_2}{t_2} - \frac{bs_2}{2}\left(\frac{1}{bt_1} + \frac{1}{at_2}\right)\right]$$

The warp at stringer 2 is obtained by setting $s_2 = a$. We have

$$w_2 = -w_1$$

Similarly, the warps along s_3, s_4 and s can be calculated using (4.98). The result would show that

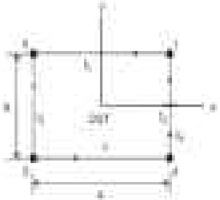
$$w_3 = w_4 = -w_1 = -w_2$$

Of course, the result above can also be deduced from the symmetry of the cross-section.

Note that a positive value of w means that the warp is in the positive z-direction if T is positive (counterclockwise).

It is obvious from (9.90) that the sign of w changes if $a/t_a = b/t_b$ and that no warping occurs for square sections of uniform wall thickness.

Find warping at 1, 2, 3 and 4. Take a= 300 mm, b= 200 mm, t_a= 8 mm, t_b =4 mm, G= 27 GPa, and T=100 kN m.



Tutorials Prob. III

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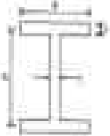


[illegible]

[illegible]

Can we compare the weights per unit length of these two beams:

The weights per unit length is defined as

$w = \rho \cdot A$, where ρ = density , and A = cross-sectional area

(i) For the Steel beam

$\rho_{st} = 7.8\,(g/cm^3) = 7.8 \times 10^{-3}\,(g/mm^3)$

$A_{st} = (200 - 5) \times 5 + 2 \times 50 \times 5 = 1475\,(mm^2)$

$w_{st} = \rho_{st} \cdot A_{st} = 7.8 \times 10^{-3} \times 1475 = 11.5\,(g/mm)$

(ii) For the Aluminum beam

$\rho_{al} = 2.70\,(g/cm^3) = 2.70 \times 10^{-3}\,(g/mm^3)$

$A_{al} = (200 - 7) \times 7 + 2 \times$ [illegible] $=$ [illegible] (mm^2)

$w_{al} = \rho_{al} \cdot A_{al} = 2.70 \times 10^{-3} \times$ [illegible] $=$ [illegible] (g/mm)

For a unit length of both materials, the aluminum beam is much lighter than the steel beam. It seems that the [illegible]

Prob. 02 [illegible]

$$[illegible] \qquad [illegible] \qquad [illegible]$$

$$[illegible] \qquad [illegible] \qquad [illegible]$$

[illegible]

Solution:

[illegible]

[illegible]

$[illegible] = \begin{bmatrix} [illegible] \\ [illegible] \\ [illegible] \end{bmatrix}$ [illegible]

$[illegible] = \begin{bmatrix} [illegible] & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] \end{bmatrix}$ [illegible]

[illegible]

and $\{n_k\}^T = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$ [illegible]

[illegible]

[illegible]

Then
$\{1/\sqrt{2}, -1/\sqrt{2}, 0\}$

[illegible]

(d) The normal component $\sigma_n = \{t\}^T \cdot \{n\}$,

$$\sigma_n = \begin{Bmatrix} \frac{6}{\sqrt{2}} & \frac{5}{\sqrt{2}} & 0 \end{Bmatrix} \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{Bmatrix} = \frac{11}{2} = 5.5\text{MPa}$$

Prob. 03

Find the principal stresses and corresponding principal directions for the stresses given in Problem 2.6. Check the result with other methods such as Mohr's circle.

Solution:

(a) The stress matrix is given by

$$[\sigma_{ij}] = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find the principal stresses, we solve for $\left|[\sigma_{ij}] - \sigma[I]\right| = 0$, or

$$\begin{vmatrix} 4-\sigma & 2 & 0 \\ 2 & 1-\sigma & 0 \\ 0 & 0 & -\sigma \end{vmatrix} = 0$$

$$\begin{vmatrix} 3-\sigma & 2 & 0 \\ 2 & 4-\sigma & 0 \\ 0 & 0 & -\sigma \end{vmatrix} = 0$$

Expanding the determinant yields $-\sigma(\sigma^2 - 7\sigma + 8) = 0$, the solutions of σ are

$\sigma = 0$, or $\sigma = \dfrac{7 \pm \sqrt{17}}{2}$ (which are 1.4384 and 5.5616)

(i) When $\sigma_1 = 0$

We have the equations

$$\begin{cases} 3n_1 + 2n_2 = 0 \\ 2n_1 + 4n_2 = 0 \\ 0 \cdot n_3 = 0 \end{cases}$$ and also we have $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$

So the solution can be obtained uniquely as

$$\begin{cases} n_1 = 0 \\ n_2 = 0 \\ n_3 = 1 \end{cases}, \text{ and } \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ is the corresponding principal direction.

(ii) When $\sigma_2 = 1.43845$

We have the equations:

$$\begin{cases} 2.56155 n_1 + 2 n_2 = 0 \\ 2 n_1 + 1.56155 n_2 = 0 \\ -1.43845 n_3 = 0 \end{cases} \quad \text{...and also we have } (n_1)^2 + (n_2)^2 + (n_3)^2 = 1$$

Therefore we have the corresponding principal direction:

$$\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.61541 \\ -0.78821 \\ 0 \end{Bmatrix}$$

(iii) When $\sigma_3 = 5.56155$

We have the equations:

$$\begin{cases} -1.56155 n_1 + 2 n_2 = 0 \\ 2 n_1 - 2.56155 n_2 = 0 \\ -5.56155 n_3 = 0 \end{cases} \quad \text{...and also we have } (n_1)^2 + (n_2)^2 + (n_3)^2 = 1$$

Therefore we have the corresponding principal direction:

$$\begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0.78821 \\ 0.61541 \\ 0 \end{Bmatrix}$$

(b) Comparing with Mohr's circle

Since the stresses associated with z are all zero, we know one principal stress is 0, and its corresponding principal direction is $\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So here we can use the 2D analysis on the x-y plane just for other principal values.

[illegible]

According to the Mohr's circle, we have the radius of the circle

$$r = \frac{1}{2}\sqrt{(4-3)^2 + (2+2)^2} = 2.0615;$$

The central coordinate of the circle is

$$(\sigma_c, \tau_c) = \left(\frac{4+3}{2}, \frac{2-2}{2}\right) = (3.5, 0)$$

Therefore we have the maximum and minimum stresses, respectively

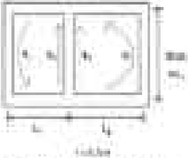
$\sigma_{max} = \sigma_c + r = 3.5 + 2.0615 = 5.5615$

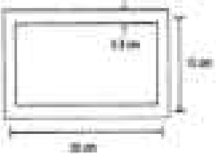
$\sigma_{min} = \sigma_c - r = 3.5 - 2.0615 = 1.4385$

These are the same as the obtained above.

[illegible]

Prob: 04





[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$q_{2} = \frac{[illegible]}{[illegible]} q_{1} \qquad (3.3.6)$$

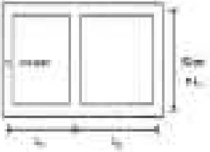
[illegible]

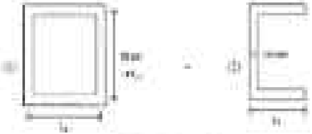
(2) Case I: $L_1 = L_2 = 10cm$ and $L_3 = 18cm$

From (3.3.6), $q_1 = q_3$, then substituting into (3.3.7) we have

$$J_{(\ldots)} = \frac{4L_1 L_2 (L_1 L_3 q_1 + L_2 L_3 q_3)^2}{(2q_1 L_3 - 2q_2 L_1 - q_3 L_2)} = 800cm^4$$

Prob. 05





Where

$$[illegible] \quad (3.6.3)$$

[illegible]

and

$$[illegible] \quad (3.6.4)$$

[illegible]

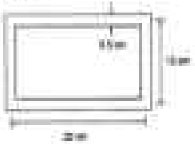
So, from (3.6.2) we get:

[illegible]

(c) The reduction of torsional rigidity is obtained as:

$$R = [illegible] = 0.427 = 42.7\%$$

Prob. 06 [illegible]



[illegible]

Solution:

[illegible]

Prob. 07 [illegible]

[illegible]

[illegible]

The twist angle of the left cell is

$$\theta_1 = \frac{1}{2GA_1}\oint \frac{q\,ds}{t} = \frac{1}{2GA_1}\left(\frac{s_1}{t_1}q_1 + \frac{s_{12}}{t_{12}}(q_1 - q_2)\right) \qquad (3.8.2)$$

where $s_1 = \frac{\pi r}{2} = 0.785\text{m}$ is the length of the left side wall, and $s_{12} = $ [illegible] is the length of the vertical wall.

The twist angle of the right cell is

$$\theta_2 = \frac{1}{2GA_2}\oint \frac{q\,ds}{t} = \frac{1}{2GA_2}\left(\frac{s_2}{t_2}q_2 - \frac{s_{12}}{t_{12}}(q_1 - q_2)\right) \qquad (3.8.3)$$

Again we have $s_2 = \frac{\pi r}{2} = 0.785\text{m}$, the length of the right side wall.

Since the entire thin-wall section must rotate as a rigid body in the plane, we require the compatibility condition

$$\theta_1 = \theta_2 = \theta = [illegible] \qquad (3.8.4)$$

From (3.8.2) to (3.8.4) and noting that $A_1 = A_2$, we derive the relation between q_1 and q_2 by substituting all the known quantities.

$$\frac{0.[illegible]}{0.[illegible]}q_1 + \frac{[illegible]}{0.[illegible]}q_{12} = \frac{0.[illegible]}{0.[illegible]}q_2 - \frac{[illegible]}{0.[illegible]}q_{12}$$

Substituting $q_{12} = q_1 - q_2$ in the equation above, we obtain

$$q_1 = 1.0[illegible]\,q_2 \qquad (3.8.5)$$

Substituting $q_2 = q_1 \cdot \gamma_1$, in the equation above, we obtain

$$q_2 = 1.657 q_1 \qquad (3.6.5)$$

Back substituting into (3.5.2) and (3.5.1), we have

$$\theta = 0.0124 \, rad/m = \frac{[illegible]}{[illegible]}$$

From which we obtain

$$q_1 = 453{,}569 \; N/m$$

Subsequently from (3.6.5) we obtain

$$q_2 = 1.657 q_1 = 751586 \; N/m$$

(iii) The applied torque

From (3.5.1), we compute the applied torque:

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 = 2([illegible] \times 453569 + [illegible] \times 751586)$$

$$= [illegible] \; N-m = 2.[illegible] \times 10^6 \; N-m$$

(iv) The torsion constant J

From the fundamental relationship of torque and twist angle, we have $T = GJ\theta$

So the torsion constant can be derived as

$$J = \frac{T}{G\theta} = \frac{[illegible]}{(27 \times 10^9)(0.0124)} = [illegible] \times 10^{-3} m^4$$

Prob. 08

Solution:

The twist angle of the left cell is

$$\theta_1 = \frac{1}{2GA_1}\oint_{cell\,1}\frac{q\,ds}{t} = \frac{1}{2GA_1}\left(\frac{s_1}{t_1}q_1 + \frac{s_{12}}{t_{12}}(q_1 - q_2)\right) \quad [illegible]$$

where $s_1 = \frac{\pi r}{2} = 1.88\,m$ is the length of the left half circular wall, and $s_{12} = 1.2\,m$ is the length of the vertical web.

The twist angle of the right cell is

$$\theta_2 = \frac{1}{2GA_2}\oint_{cell\,2}\frac{q\,ds}{t} = \frac{1}{2GA_2}\left(\frac{s_2}{t_2}q_2 + \frac{s_3}{t_3}q_2 - \frac{s_{12}}{t_{12}}(q_1 - q_2)\right)$$

Again, we have $s_2 = 2\,m$, the length of the lower wall,

and $s_3 = \sqrt{2^2 + 1.2^2} = 2.33\,m$, the length of the inclined wall of thickness t_3 in the right cell.

Since the entire thin-wall section must rotate as a rigid body in the plane, we require the compatibility condition

$$\theta_1 = \theta_2 = \theta \quad [illegible]$$

Solving the two equations [illegible], we obtain

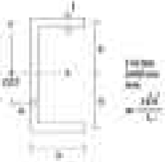
[illegible]

Eliminating [illegible] from the equation [illegible], we obtain

$$q_1 = 1.17q_2 \quad [illegible]$$

Prob. 09

Prob. 10



[illegible]

(b) For the channel section in Fig. 1-40, assume the shear modulus is G. We can first derive some needed properties.

The moment of inertia with respect to x axis is

[illegible]

(Since the thickness of walls is relatively small, there are some approximated solutions such as [illegible] or [illegible] are all the acceptable approximations)

The eccentric distance [illegible]

The moment of inertia is

$$[illegible]$$

and the twist per unit length can be obtained from

$$[illegible] \qquad (1)$$

(ii) Dividing the section into two straight parts s_1 and s_2 as shown below

[illegible]

For contour s_1 we have

$$[illegible] \qquad (1.14.7)$$

and for contour s_2 (The point $s_2 = 0$ is at upper left corner of the section) we have

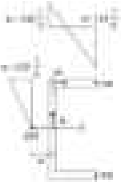
$$[illegible] \qquad (1.14.8)$$

(iii) For the contour s_1 the warping displacement w is calculated from equation (1.14.2):

$$[illegible]$$

In which the condition $w(s_1=0)$ has been used. This is obvious since the warping at the middle point of the vertical web is zero because of anti-symmetry. Also, we have

[illegible]



Bending and Flexural Shear

Bending is the most frequently occurring form of load in aircraft structures.

An analysis becomes very cumbersome and often intractable when the loading becomes complicated.

For thin-walled sections such as boxed-beam section or multichannel section, analytical solutions are not possible even for simpler loadings.

When loadings are only lateral and deformation is not large, a simpler inverse approach, called Bernoulli-Euler beam theory is used.

The main disadvantage of the Bernoulli-Euler beam theory is that this theory is not applicable when transverse shear deformation is important.

In such a situation, the Timoshenko beam theory is appropriate.

For excessive bending deformation, other methods such as inelastic bending or plastic bending approaches are often used. However, when both the section geometries and loadings get complicated, none of the analytical methods will work.

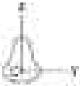
In such situations, numerical methods such as finite element and finite difference methods need to be used.

BERNOULLI-EULER BEAM THEORY

Consider a straight beam (bar) of a uniform cross-section that is symmetrical with respect to a vertical line.

Straight beam of a uniform and symmetrical cross-section

If the width of the beam is small, then the state of stress due to transverse loading can be approximated by **plane stress** parallel to the x-z plane, and u and w can be assumed to be functions of x and z only.



Expand u and w in power series of z as:

$$u(x,z) = u_0(x) + z u_1(x) + z^2 u_2(x) + \cdots \quad (3.1)$$

$$w(x,z) = w_0(x) + z w_1(x) + z^2 w_2(x) + \cdots \quad (3.2)$$

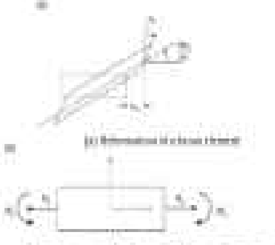
For slender beams, the depth is small compared with the length. In other words, the range of z is small, and the high-order terms in z make insignificant contributions.

Hence, as a first-order approximation, we truncate the series of Previous in

$$u = u_0(x) + z\psi_y(x)$$

$$w = w_0(x)$$

Figure 3.2



(a) Deformation of a beam element

(b) sign convention for the resultant force and moments

From Eq. (5.3), it is obvious that u_0 represents the longitudinal displacement of the centroidal axis, and ϕ_y represents the rotation of the cross-section after deformation; see Figure 5.2a. From Eq. (5.3), a positive rotation ϕ_y is clockwise, which is opposite to the slope dw_0/dx of the beam deflection.

Note that $u(x, z)$ is a linear function of z. This implies that **plane cross-sections remain plane after deformation** but may not be perpendicular to the centroidal axis.

The strain components corresponding to the approximate displacements given by Eqs. (5.3) and (5.4) are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} + z\frac{d\phi_y}{dx} \qquad (5.5)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{dw_0}{dx} + \phi_y \qquad (5.6)$$

Define the resultant axial force N_x and bending moment M_y as

$$N_x = \iint_A \sigma_{xx}\, dA \qquad (3.7)$$

$$M_y = \iint_A z\sigma_{xx}\, dA \qquad (3.8)$$

in which the area integration is over the entire cross-section.

For slender beams, the transverse shear strain γ_{xz} is small. In calculating the bending strain, we can assume that $\gamma_{xz} = 0$ as an approximation. This leads to, from Eq (3.6),

$$\psi_y = -\frac{dw_0}{dx} \qquad (3.9)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{dw_0}{dx} + \psi_y$$

The relation above implies that the plane cross section remains perpendicular to the centroidal axis after deformation, and that the amount of rotation of the cross section is equal to the slope of deflection.

Using Eqs. (5.6) and (5.3), we obtain

$$\varepsilon_{xx} = \frac{du}{dx} = \frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}$$

$$\sigma_{xx} = E\varepsilon_{xx} = E\left[\frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}\right] \quad (5.10)$$

Substitution of Eq. (5.10) into Eqs. (5.7) and (5.8) yields

$$N_x = EA\frac{du_0}{dx} - E\frac{d^2 w_0}{dx^2}\iint_A z\,dA \quad (5.11)$$

$$M_y = E\frac{du_0}{dx}\iint_A z\,dA - E\frac{d^2 w_0}{dx^2}\iint_A z^2\,dA \quad (5.12)$$

Since the origin of the coordinates coincides with the centroid of the cross-section, we have

$$\iint_A z\,dA = 0$$

Thus, Eqs. (5.11) and (5.12) reduce to

$$N_x = EA\frac{du_0}{dx} \quad (5.14)$$

$$M_y = -EI_y\frac{d^2 w_0}{dx^2} \quad (5.15) \quad \text{where} \quad I_y = \iint_A z^2\,dA$$

If no axial force is applied, i.e. $N_1 = 0$, then $d\varepsilon_0/dx = 0$. From Eq. (5.5), this means that $\varepsilon_{11} = 0$ along the x_1 axis (or, more precisely in the x_1-y plane). Thus, the x_1 axis is the neutral axis and the x_1-y plane is the neutral plane.

After adopting the approximation of Eq. (5.4), we note that $\gamma_{12} = 0$, and, as a result, the transverse shear stress τ_{12} cannot be obtained from the shear strain (which is approximated to be zero).

The resultant transverse shear force

$$V_2 = \iint_A \tau_{12} \, dA$$

should be obtained from considering the equilibrium of a beam element as shown in Figure 5.3.

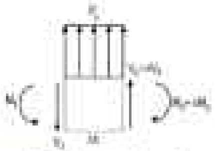


Fig. 5.3 Equilibrium of a beam element.

The force equilibrium in the x_2-direction gives

$$\Delta V_2 + p_2 \Delta x = 0$$

Taking the limit $\Delta x \to 0$, we have

$$\frac{dV_z}{dx} = -p_z \qquad (5.17)$$

The equilibrium of moments about the y-axis located at the left end of the beam element in Figure 5.2 yields

$$-\Delta M_y + p_z \cdot \Delta x \cdot \tfrac{1}{2}\Delta x + (V_z + \Delta V_z)\Delta x = 0$$

After taking $\Delta x \to 0$ (and thus $\Delta V_z \to 0$), we obtain

$$\frac{dM_y}{dx} = V_z \qquad (5.18)$$

It is evident that the transverse shear force V can be derived from the bending moment.

This is satisfied by setting $\tau_{xz} = 0$ (see Eq. (5.14)). Evidently, the assumption $\gamma_{xz} = 0$ is exact in this case when moment is constant along the beam.

Substituting Eq. (5.15) into Eq. (5.18) and then into Eq. (5.17), we obtain

$$EI_y \frac{d^4 w_0}{dx^4} = p_z$$

This is the **Bernoulli–Euler beam** (classic beam) **equation**.

In the absence of axial force, i.e. $N = 0$, we have $du_0/dx = 0$ from Eq. (5.14), and the bending strain relation is

$$\varepsilon_{xx} = -z\frac{d^2 w_0}{dx^2} \qquad (5.20)$$

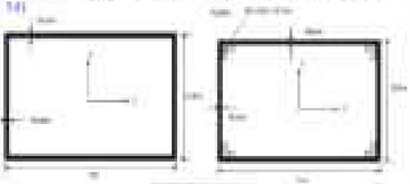
[illegible]

Using Eqs. (5.15) and (5.20), we can write:

$$\varepsilon_{xx} = \frac{M_y z}{E I_y} \qquad \sigma_{xx} = \frac{M_y z}{I_y}$$

Example

[illegible] (Figure [illegible])



$$I_z = \frac{0.1(0.6)^3}{12} - \frac{0.08(0.58)^3}{12} = 2.1858 \times 10^{-3}\ m^4$$

The direct stress distribution is given by

$$\sigma_{xx} = \frac{M_z}{I_z} y = \frac{-100 \times 10^{3}}{2.1858 \times 10^{-3}} y = -48.41 y\ MPa$$

On the top and bottom sections, $y = \pm 0.3$ m. As such,

$$\sigma_{xx} = -14.52\ \text{MPa at top of the section}$$

$$\sigma_{xx} = 14.52\ \text{MPa at bottom of the section}$$

[illegible]

b. This section is also symmetric [illegible]. The moment of inertia I_z including the angles can be calculated as follows:

$$I_z = \left(\frac{[illegible]}{12} - \frac{[illegible]}{12}\right) + 4\left[\frac{[illegible]}{12} + [illegible]\right]$$

$$= \left[\frac{[illegible]}{12} + [illegible]\right] = 2.659 \times 10^{-3}\ m^4$$

The direct stress distribution is given by

$$\sigma_{xx} = \frac{M_z}{I_z} y = \frac{-100 \times 10^{3}}{2.659 \times 10^{-3}} y = -37.61 y\ MPa$$

[illegible]

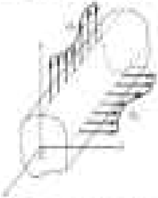
So the top and bottom sections, $z = \pm 0.5$ m, do reach,

$$\sigma_x = -11.20 \text{ MPa at top of the section}$$

$$\sigma_x = 11.20 \text{ MPa at bottom of the section}$$

The addition of the angles lowers the direct stresses on the section. In other words, load carrying capacity of a section can be improved by adding stringers/angles.

Bidirectional Bending on Beams with an Arbitrary Section



Beam with an arbitrary cross section under bidirectional bending

For beams with arbitrarily shaped cross-sections, we set up the coordinate system as shown in Figure.

Again, the x-axis is chosen to coincide with the centroidal axis.

The external load is decomposed into p_y and p_z in the y and z directions, respectively.

It is noted that the line loads must pass through the center of twist (shear center) if torsion is to be avoided.

Under such bidirectional bending, the longitudinal displacement is a function of x, y, and z.

The approximate displacement expansions similar to Eqs. (5.3) and (5.4) are given by

$$u = u_0(x) + z\psi_y(x) - y\psi_z(x)$$

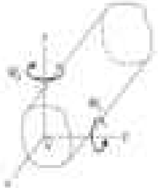
$$v = v_0(x)$$

$$w = w_0(x) \qquad (5.29)$$

where ψ_y and ψ_z are rotations of the cross-section about the y and z axes, respectively.

The corresponding strains are

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{du_0}{dx} + z\frac{d\theta_y}{dx} + y\frac{d\theta_z}{dx} \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{dv_0}{dx} \pm \theta_z \qquad\qquad (5.24) \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{dw_0}{dx} + \theta_y
\end{aligned}
$$



Sign convention of M_y and M_z

Again, the simplifying assumption $\gamma_{xy} = \gamma_{xz} = 0$ yields the relations

$$\psi_z = -\frac{dv_0}{dx}$$

$$\psi_y = -\frac{dw_0}{dx}$$

which are substituted into Eq. (5.24a) to obtain

$$\varepsilon_{xx} = \frac{du_0}{dx} - y\frac{d^2 v_0}{dx^2} - z\frac{d^2 w_0}{dx^2}$$

Using the argument that $du_0/dx = 0$ (if $N_1 = 0$), the bending strain is reduced to

$$\varepsilon_{xx} = -y\frac{d^2 v_0}{dx^2} - z\frac{d^2 w_0}{dx^2} \qquad (5.26)$$

The bending moments about the y and z axes, respectively, are defined as

$$M_z = \iint_A y\sigma_{xx}\,dA = -E\iint_A \left[y^2\frac{d^2 v_0}{dx^2} + yz\frac{d^2 w_0}{dx^2}\right] dA$$

$$= -EI_z\frac{d^2 v_0}{dx^2} - EI_{yz}\frac{d^2 w_0}{dx^2} \qquad (5.27)$$

$$M_y = \iint_A z\sigma_{xx}\,dA = -EI_{yz}\frac{d^2 v_0}{dx^2} - EI_y\frac{d^2 w_0}{dx^2} \qquad (5.28)$$

Where $I_y = \iint_A z^2 \, dA$ moment of inertia about y axis

$I_z = \iint_A y^2 \, dA$ moment of inertia about z axis

$I_{yz} = \iint_A yz \, dA$ product of inertia

Solving Eq. (5.27) and (5.28), we obtain

$$-E\frac{d^2 v_0}{dx^2} = \frac{1}{I_y I_z - I_{yz}^2}(-I_{yz} M_y + I_y M_z)$$

$$-E\frac{d^2 w_0}{dx^2} = \frac{1}{I_y I_z - I_{yz}^2}(I_z M_y - I_{yz} M_z)$$

Using Eq. (5.26), we write the bending stress as

$$\sigma_{xx} = E\varepsilon_{xx} = -yE\frac{d^2 v_0}{dx^2} - zE\frac{d^2 w_0}{dx^2}$$

$$= \frac{I_y M_z - I_{yz} M_y}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y - I_{yz} M_z}{I_y I_z - I_{yz}^2} z \qquad (5.30)$$

The location of the **neutral axis** (neutral plane) along which $\sigma_{xx} = 0$ can be found from Eq. (5.30), i.e.

$$\frac{I_y M_z - I_{yz} M_y}{I_y I_z - I_{yz}^2} y = -\frac{I_z M_y - I_{yz} M_z}{I_y I_z - I_{yz}^2} z$$

Defining the neutral plane by angle α as shown in Figure 5.11, we have

$$\tan\alpha = -\frac{z}{y} = \frac{I_z M_y - I_{yz} M_z}{I_y M_z - I_{yz} M_y}$$

Note that the positive direction of angle α is clockwise.

If the y- or z-axis is an axis of symmetry for the cross-section, then $I_{yz} = 0$ and Eq. (5.20) reduces to

$$\sigma_{xx} = \frac{M_z}{I_z} y + \frac{M_y}{I_y} z$$

Further, if $M_z = 0$, the bending stress becomes

$$\sigma_{xx} = \frac{M_y}{I_y} z \qquad (5.21)$$

If $I_{yz} \neq 0$ and $M_z = 0$, then from Eq. (5.30) we have

$$\sigma_{xx} = -\frac{I_{yz} M_y}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y}{I_y I_z - I_{yz}^2} z \qquad (5.34d)$$

Therefore, for beams with an arbitrary cross-section under one-way bending, say $M_y \neq 0$ and $M_z = 0$, the simple beam bending stress formula (5.33) is not valid, and Eq. (5.34d) must be used.

For $M_z \neq 0$ and $M_y = 0$, the corresponding bending stress equation can be obtained as

$$\sigma_{xx} = -\frac{I_y M_z}{I_y I_z - I_{yz}^2} y + \frac{I_{yz} M_z}{I_y I_z - I_{yz}^2} z$$

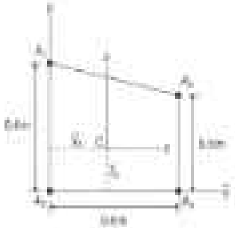
the displacement equilibrium equations

$$EI_{zz} \frac{d^4 v_0}{dx^4} + EI_{yz} \frac{d^4 w_0}{dx^4} = p_y$$

$$EI_{yz} \frac{d^4 v_0}{dx^4} + EI_{yy} \frac{d^4 w_0}{dx^4} = p_z$$

Example

The cross-section of a single-cell box beam with four stringers is shown in Figure 5.14. The contribution of the thin sheets to bending is assumed to be negligible. Thus, only the areas of the stringers are considered in the bending analysis. Using structural idealization method, the areas of the stringers are estimated as $A_1 = 6 \times 10^{-4} m^2$, $A_2 = 5 \times 10^{-4} m^2$, $A_3 = A_4 = 4 \times 10^{-4} m^2$. Find the bending stress distribution.



Single-cell box beam with four stringers

First, the centroid of the effective cross-sectional area (i.e., those of the stringers) must be determined. Denoting the coordinates of the stringers by $(\bar{y}_i, \bar{z}_i)$ with respect to the $\bar{y}$-$\bar{z}$ system, we have the coordinates of the centroid

$$\bar{y}_c = \frac{\sum \bar{y}_i A_i}{\sum A_i} = \frac{0A_1 + 0A_2 + 0.5A_3 + 0.5A_4}{A_1 + A_2 + A_3 + A_4} = \frac{4 \times 10^{-4}}{(8 \times 10^{-4})} = 0.21\ \text{m}$$

$$\bar{z}_c = \frac{\sum \bar{z}_i A_i}{\sum A_i} = \frac{0.4A_1 + 0A_2 + 0A_3 + 0.3A_4}{A_1 + A_2 + A_3 + A_4} = \frac{3.8}{18} = 0.19\ \text{m}$$

Thus, the location of the centroid is (0.21, 0.19) in the system.

This cross-section consisting of four stringers is not symmetric with respect to either the y- or the z-axis. Hence, the general bending equations must be used.

The moments of inertia of the effective cross-sectional area of the box beam with respect to the coordinate system y–z are calculated.

Denoting the coordinates of each stringer by (y_i, z_i) with respect to the y–z system, we have

$$I_y = \sum A z^2$$

$$= [illegible]$$

$$= 3.61 \times 10^{[illegible]} \, m^4$$

$$I_z = \sum A y^2$$

$$= [illegible] = 1.16 \times 10^{-4} \, m^4$$

$$I_{yz} = \sum A y z$$

$$= [illegible]$$

$$= -0.15 \times 10^{-4} \, m^4$$

Consider the loading $M_y = 0$ and $M_z \neq 0$. The neutral plane is given by

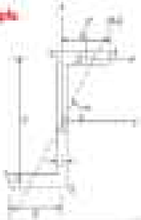
$$\tan\alpha = -\frac{I_{yz} M_z}{I_y M_z} = \frac{0.15 \times 10^{-4}}{1.16 \times 10^{-4}} = 0.13$$

which yields $\alpha = 7°$ measured clockwise from the y axis.

The bending stresses in the stringers can be calculated easily using Eq (7.22). [illegible]

A beam of the thin-walled Z-section shown in Figure is subjected to a positive bending moment M_z. Find the distribution of bending stresses.

Example



In this case the centroid is easy to locate to be at the midpoint of the vertical web as shown in Figure.

The moment of inertia of the cross-section is the sum of the moments of inertia of the three rectangular sections of the web and two flanges. For each section it is most convenient to calculate I_y, I_z and I_{yz} using the parallel axis method.

For example, consider the upper flange 3-4. Let the y' and z' axes be local coordinates with the origin at the centroid C' of the upper flange.

According to the parallel axis theorem, the moment of inertia I_y of the cross-section of the upper flange is

$$I_y = A_{34}\left(\frac{h}{2}\right)^2 + I_{y'}$$

in which A_{fl} is the cross-sectional area of the upper flange and $I_{y'}$ is the moment of inertia about y'-axis.

For the upper flange, we have

$$I_{y'} = \frac{th^3}{12}$$

Similarly, the moment of inertia about the z-axis can be obtained in a similar manner.

The product of inertia of the upper flange can be written as:

$$I_{yz} = A_{fl}\frac{h}{2}\frac{b}{2} + I_{y'z'}$$

and $I_{y'z'} = 0$ because of symmetry with respect to the local coordinate system.

For the entire Z-section, we obtain

$$I_y = 2bt\left(\frac{h}{2}\right)^2 + \frac{2tb^3}{12} + \frac{th^3}{12}$$

$$I_z = 2bt\left(\frac{b}{2}\right)^2 + \frac{2tb^3}{12} + \frac{ht^3}{12}$$

$$I_{yz} = bt\frac{h}{2}\frac{b}{2} + bt\left(-\frac{h}{2}\right)\left(-\frac{b}{2}\right) = \frac{h^2bt}{2}$$

For thin-walled sections, the terms with t are small and are neglected in the following calculations.

The orientation of the neutral axis can be calculated using Eq. (5.21) with $M_y = 0$. Thus

$$\tan\alpha = -\frac{I_{yz}}{I_z} = -\frac{3h}{4b}$$

Thus, $\alpha = -\tan^{-1}(3h/4b)$. Note that a negative value of α means that the neutral axis is oriented at a counterclockwise rotation of an angle $-\alpha$ from the y-axis as shown in Figure 5.18.

Let us consider the case with $h = 2b$, which leads to $\alpha = -\tan^{-1} 1.5 = -56.3^\circ$.

Note that in this case, points 3 and 4 on the upper flange are on opposite sides of the neutral axis, and the corresponding bending stresses must be of opposite signs.

At the free edge of the upper flange (point 4 in Figure 5.18) the bending stress is calculated using Eq. (5.24a) together with $y = h/2$, $z = b/2$. We have, for a positive bending moment,

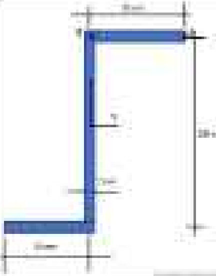
$$\sigma_4 = -\frac{1.713M_z}{th^2}$$

(compression)

Similarly, at point 3 ($y = h/2$, $z = -b/2$), we obtain

$$\sigma_3 = \frac{1.428M_z}{th^2}$$

(tension)



Find the location of the Principal axes and calculate the axial stress at location A and B. Take M_x = 1.80 kNm; M_y = 0

Answer:

I_x = 666.73 X10^3 mm^4

I_y = 166.73 X10^3 mm^4

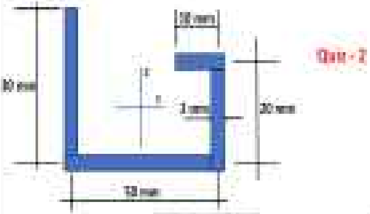
I_{xy} = 250 X10^3 mm^4

α = -58.3°

σ_{zz} @ A = 4.57 MPa

σ_{zz} @ B = 17.13 MPa

Find the position of the principal planes for the given unsymmetrical beam subjected to bending moments, M_x = 1900 Nm and M_y = 150 Nm.



Example

Figure shows the cross-section of a four-stringer box beam. Assume that the thin walls are ineffective in bending and the applied bending moments are $M_x = -500{,}000$ N.cm, $M_y = 200{,}000$ N.cm.

Find the bending stresses in all stringers.

(a) Set up a (temporary) coordinate system with stringer 1 as the origin. The location of the centroid is:

$$[illegible]$$

$$[illegible]$$

(b) The moment of inertia:

$$I_x = \sum A_i \bar{y}_i^2 = \text{[illegible]}$$

$= \text{[illegible]}\ cm^4$

Similarly

$$I_y = \sum A_i \bar{x}_i^2 = \text{[illegible]}\ cm^4$$

$$I_{xy} = \sum A_i \bar{x}_i \bar{y}_i = \text{[illegible]}\ cm^4$$

[illegible]

Stringer No.	A (cm^2)	$\bar{x}$ (cm)	$\bar{y}$ (cm)	$A\bar{x}^2$ (cm^4)	$A\bar{y}^2$ (cm^4)	$A\bar{x}\bar{y}$ (cm^4)
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
			$\sum$	[illegible]	[illegible]	[illegible]

(c) Bending stress at the stringers

By using the equation $\sigma_{xx} = \frac{I_z M_y - I_{yz} M_z}{I_y I_z - I_{yz}^2} z + \frac{I_y M_z - I_{yz} M_y}{I_y I_z - I_{yz}^2} y$, and

$M_y = $ [illegible]

$M_z = $ [illegible]

$I_y = $ [illegible]

$I_z = $ [illegible]

$I_{yz} = $ [illegible]

Therefore the bending stresses at the stringers are:

Stringer No.	y (mm)	z (mm)	σ_{xx} (N/mm²)
1	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]
3	[illegible]	[illegible]	[illegible]
4	[illegible]	[illegible]	[illegible]

We obtain: $\sigma_{xx} = -1.298y - 21.538z$

Neutral axis in the thin walled section

$$\sigma_{xx} = -1.298y - 21.538z = 0$$

On the cross-section, this equation represents the line passing through the centroid with $y = -16.59z$ and an angle

$$\alpha = \tan^{-1}\left(-\frac{z}{y}\right) = \tan^{-1}\left(\frac{1}{16.59}\right) = 3.45°$$

Find the bending stresses in the stringers at the fixed end of the box beam loaded as shown in Figure. Assume that the thin sheets are negligible in bending. Find the neutral axis.

Example

Loaded box beam

Solution:

(a) Name the stringers from top to bottom and left to right as stringer 1, stringer 2, and stringer 3, respectively. Relative to string 2 the centroid position is given by

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{4 \times 80}{3 \times 4} = 26.67\,cm$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{4 \times 40}{3 \times 4} = 13.33\,cm$$

(b) The bending moments at the fixed end of the bar (beam) produced by the loads are

$M_x = [illegible]$ (positive in positive x)

$M_y = [illegible]$ (positive in negative y)

(c) Set up the coordinate system (x, y, z) with the origin at the centroid.

Moment of inertia (see table below for details):

$I_x = \sum A_i y_i^2 = [illegible]$

$I_y = \sum A_i x_i^2 = [illegible]$

$I_{xy} = \sum A_i x_i y_i = [illegible]$

Stringer No.	A (cm²)	x_i (cm)	y_i (cm)	Ax_i^2 (cm⁴)	Ay_i^2 (cm⁴)	Ax_iy_i (cm⁴)
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
			$\sum =$	[illegible]	[illegible]	[illegible]

(d) Bending stress in the stringers

Using the equation $\sigma_z = \frac{I_x M_y - I_{xy} M_x}{I_x I_y - I_{xy}^2} x + \frac{I_y M_x - I_{xy} M_y}{I_x I_y - I_{xy}^2} y$

we obtain: $\sigma_{zz} = -31.25y - 78.125x$

and the bending stresses in the stringers are:

Stringer No.	x (cm)	y (cm)	σ_z (T/m^2)
1	-20.07	22.07	-1200
2	-20.07	-11.33	1979
3	15.33	-11.33	420

(e) Neutral plane for angle α

Neutral plane is located at the position where bending stresses vanish under this particular loading. We have

$\sigma_{zz} = -31.25y - 78.125x = 0$

It is the line passing through the centroid with $y = -2.5x$

$\alpha = \tan^{-1}\left(-\frac{x}{y}\right) = \tan^{-1}\left(\frac{1}{2.5}\right) = 21.8°$

Find the deflection of the box beam using the simple beam theory.

The governing equations (see p. 122 in the book) for the bidirectional bending are

$$E\frac{d^2v}{dz^2} = -\frac{I_y M_z - I_{yz} M_y}{I_y I_z - I_{yz}^2} = 0.063(500 - z) \ (N/cm^2),$$

$$E\frac{d^2w}{dz^2} = \frac{I_z M_y - I_{yz} M_z}{I_y I_z - I_{yz}^2} = 0.156(500 - z) \ (N/cm^2).$$

Integrating twice the above differential equations, we obtain

$$Ev = 0.063\left(250z^2 - \frac{z^3}{6}\right) + C_1 z + C_2$$

$$Ew = 0.156\left(250z^2 - \frac{z^3}{6}\right) + C_3 z + C_4$$

By applying the boundary conditions, the integration constants are solved as

$$v(z=0) = 0, \ \frac{dv}{dz}(z=0) = 0 \implies C_1 = C_2 = 0$$

$$w(z=0) = 0, \ \frac{dw}{dz}(z=0) = 0 \implies C_3 = C_4 = 0$$

Then the lateral (in y-direction) and vertical (in z-direction) deflections are, respectively:

$$v(x) = \frac{0.065}{E}\left(250x^2 - \frac{x^3}{6}\right)$$

$$w(x) = \frac{0.156}{E}\left(250x^2 - \frac{x^3}{6}\right)$$

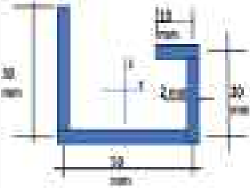
In the expressions above, distance x is measured in cm, and the units of Young's modulus and deflection are N/cm^2 and cm, respectively.

As an example, consider Aluminum 2024-T3 ($E = 72GPa = 72\times10^5 N/cm^2$).

The deflections in y and z directions at the free end are:

$$v(x = 500) = \frac{0.065}{72\times10^5}\times\left(250\times500^2 - \frac{500^3}{6}\right) = 0.38cm$$

$$w(x = 500) = \frac{0.156}{72\times10^5}\times\left(250\times500^2 - \frac{500^3}{6}\right) = 0.90cm$$



Quiz-2

Answer

- Y_c = 25.55 mm
- Z_c = 8.72 mm
- I_y = 18.298309 mm⁴
- I_z = 78.713303 mm⁴
- I_{yz} = -3.6988103 mm⁴
- α = -57.9°

Quiz 2 question

Find

σ_{xx} at A, B and C and Principal Angle

M_y = 1000 N mm

thickness 2 mm

Centroid

Y = 29.78 mm Z = 29.78 mm

And

I_y = 194808 mm4

I_z = 404808 mm4

I_{yz} = -168080 mm4

σ_{xx} at A = 0.2341 MPa

B = -0.1523 MPa

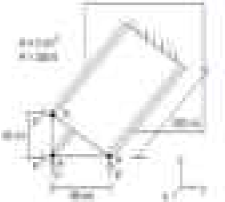
C = 0.09508 MPa

Alpha = 30 degree

Find the bending stresses in the stringers of the box beam in Fig. for the bending moments given in previous problem.

$M_x = -500{,}000$ N·cm

$M_y = 200{,}000$ N·cm



Solution:

(a) Name the stringers from top to bottom and then left to right as stringer 1, stringer 2, and stringer 3, respectively. The centroid position is given by

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = [illegible]$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = [illegible]$$

relative to stringer 2

(b) Moment of inertia (see the table below for details)

$$I_y = \sum_i A_i z_i^2 = 4(2 \times 13.33^2 + 26.67^2) = 4267 cm^4$$

$$I_z = \sum_i A_i y_i^2 = 4(2 \times 26.666667^2 + 53.333333^2) = 17067 cm^4$$

$$I_{yz} = \sum_i A_i y_i z_i = -4267 cm^4$$

Stringer No.	A_i (cm²)	y_i (cm)	z_i (cm)	$A_i z_i^2$ (cm⁴)	$A_i y_i^2$ (cm⁴)	$A_i y_i z_i$ (cm⁴)
1	4	-26.67	26.67	2844	2844	-2844
2	4	-26.67	-13.33	711	2844	1422
3	4	53.33	-13.33	711	11378	-2844
			$\sum =$	4267	17067	-4267

(c) Bending stress in the stringers.
Substituting the moments and moments of inertia in the bending stress formula

$$\sigma_z = \frac{I_{xx} M_y - I_{xy} M_x}{I_{xx} I_{yy} - I_{xy}^2}\, x + \frac{I_{yy} M_x - I_{xy} M_y}{I_{xx} I_{yy} - I_{xy}^2}\, y$$

we obtain $\sigma_z = -[illegible]$

Therefore, the bending stresses in the stringers are

Stringer No.	x (mm)	y (mm)	σ_z (N/mm^2)
1	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]
3	[illegible]	[illegible]	[illegible]

TRANSVERSE SHEAR STRESS DUE TO TRANSVERSE FORCE IN SYMMETRIC SECTIONS

In deriving the Bernoulli-Euler beam equation, the transverse shear strain γ_{xz} was neglected while the transverse shear stress τ_{xz} (and, thus, the transverse shear force V_z) was kept in the equilibrium equation. Such contradictions are often found in simplified structural theories.

The assumption $\gamma_{xz} = 0$ is quite good for slender beams (i.e. the depth is small compared with the span). In fact, it is exact if the loading is a pure bending moment.

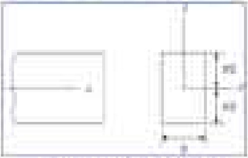
However, for short beams under transverse loads, significant shear strain (strain) may result.

The exact distribution of τ_{xz} on the cross-section of a beam subjected to transverse forces is generally not easy to analyze. An exception is the narrow rectangular section as shown in the following Figure.

If $h >> b$, the plane stress assumption adopted in the derivation of the simple beam theory is valid. In other words, τ_{xz} can be assumed to be uniform across the width of the section. Otherwise, τ_{xz} is a function of y.

Narrow Rectangular Cross Section

Consider a beam with a narrow rectangular cross-section as shown in Figure 5.16. The resultant transverse shear force V_z is



Narrow rectangular section.

$$V_z = b\int_{-h/2}^{h/2} \tau_{xz}\,dz$$

This definition alone is not sufficient to recover the distribution of τ_{xz} in the vertical (z) direction. We resort to the equilibrium equation for a state of plane stress parallel to the x-z plane (see Eq. (2.1.1)):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (5.40)$$

Substituting Eq. (5.22) into Eq. (5.40), we obtain:

$$\frac{z}{I_y}\frac{dM_y}{dx} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\sigma_{xx} = \frac{M_y z}{I_y} \quad (5.22)$$

[illegible]

Using the relation: $\frac{dM_y}{dx} = V_z$

$$\frac{z}{I_y}\frac{dM_y}{dx} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{z V_z}{I_y}$$

Integrating the above equation from $z = -h/2$ to z, we obtain:

$$\tau_{xz}(z) - \tau_{xz}\left(-\frac{h}{2}\right) = -\frac{V_z}{2I_y}\left(z^2 - \frac{h^2}{4}\right) \qquad (5.48)$$

Since the shear stress vanishes at the top and bottom faces, i.e.

[illegible]

$$\tau_{xz}(\pm h/2) = 0$$

Equation (5.48) reduces to:

$$\tau_{xz} = \frac{V_z c^2}{2I_y}\left(1 - \frac{z^2}{c^2}\right) \quad (5.49)$$

where $c = \frac{h}{2}$

From Eq. (5.49), it is evident that τ_{xz} has a parabolic distribution over z, and the maximum value that occurs at z = 0 is

$$(\tau_{xz})_{max} = \frac{V_z c^2}{2I_y}$$

General Symmetric Sections

For uniform beams with general symmetric (with respect to the z-axis) cross sections, the simple beam results are valid, i.e.

$$\sigma_{xx} = \frac{M_y z}{I_y}$$

$$\sigma_{xx} = \frac{M_z y}{I_z}$$

$$V_z = \frac{dM_y}{dx}$$

However, the transverse shear stress τ_{xz} distribution over the cross section is difficult to analyze. For symmetrical sections under a transverse shear force V_z the only thing we know is that the distribution of τ_{xz} is symmetrical with respect to the z-axis.

Since the variation of τ_{xz} in the y-direction is unknown, it is more convenient to consider the transverse shear flow q_z defined as:

$$q_z(z) = \int \tau_{xz} \, dy$$

If τ_{xz} is uniform across the width t, then

$$q_z = t\tau_{xz}$$

If τ_{xz} is not uniform in the y-direction, the average value is introduced as

$$\bar{\tau}_{xz} = \frac{q_z}{t}$$

The transverse shear flow q_z can be determined from the equilibrium of a differential beam element as shown in Figure 5.17. In Figure 5.17a, the side view of the beam element of length Δx is shown with the bending stresses acting on the two neighboring cross-sections.

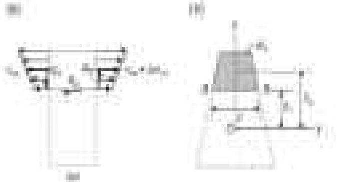


Fig. 5.17 Differential beam element with bending stresses: (a) longitudinal section, (b) cross-section.

Consider the free body of the beam element above the D-D plane (i.e., $z \geq z_1$) as shown in Figure 5.17b. The shear flow q_z on the cross-section at $z = z_1$ is equal to the shear force acting on the bottom face of the free body as depicted in Figure 5.18a. For the free body above $z = z_1$, equilibrium equation $\Sigma F = 0$ is given by: Fig.

$$\iint_{A_1} \Delta\sigma_{xx}\, dA = q_z \Delta x$$

where A_1 is the cross-sectional area above $z = z_1$. Dividing both sides of (4-40) by Δx and taking the limit $\Delta x \to 0$, we have:

$$\iint_{A_1} \frac{d\sigma_x}{dx}\, dA = q_x \qquad ([illegible])$$

By using Eq. ([illegible]) in Eq. ([illegible]), we obtain the transverse shear flow as

$$q_x = \iint_{A_1} \frac{dM_z}{dx}\frac{y}{I_z}\, dA$$

$$= \frac{dM_z}{dx}\frac{1}{I_z}\iint_{A_1} y\, dA$$

$$= \frac{VQ}{I_z}$$

Where

$$Q = \iint_{A_1} y\, dA$$

Thin Walled Sections

(a) Wide-flange beam (b) distribution of [illegible] (c) distribution of shear flow q

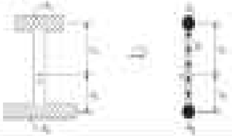
Figure 5.38

Consider the wide-flange beam shown in Figure 5.10a. The transverse shear stress is usually represented by the average value as shown in Figure 5.10b.

A jump in τ is noted at plane CDEF due to the sudden change of width. If the transverse shear flow q_y is plotted as shown in Figure 5.10c, then no such jump exists.

In the free hang portion of the wide flange (e.g., the portions CD and EF), the actual transverse shear stress τ_{xy} is much smaller than the average value. It is noted that τ_{xy} must vanish along AB, CD, and EF. If the thickness of the flange is small, then τ_{xy} cannot build up significantly except for the portion connected to the vertical web. A more accurate distribution of τ_{xy} along CDEF is depicted in following figure.

From Figure 5.18, it is seen that for wide-flange beams, the transverse shear stress is small in the flange and that the web carries the majority of the transverse shear load. An approximate model for such a wide flange beam is obtained by lumping the total area of the flange into a concentrated area as shown in Figure 5.18. In addition, we may assume that the web does not contribute to resisting bending.



Thus, for the web section we have

$$Q = A_f y_1$$

which remains unchanged with location. As a result, Q_f is constant along the web.

In aircraft structures, stringers are often used to provide bending stiffness, and thin webs are used to carry shear flows. To maximize the bending capacity of the structure, we place stringers at the greatest distance from the neutral axis.

The thin web is usually assumed ineffective in bending. Consequently the shear flow in the web between two adjacent stringers is constant.

It should be noted that although the transverse shear stress τ_{yz} is very small in the flanges, the in-plane shear stress τ_{xz} is significantly large, as discussed in this chapter. However, the existence of the in-plane shear stress does not affect the simplified calculation above of the transverse shear flow in the web.

Shear Deformation in Thin Walled Sections

In developing the simple beam theory the transverse shear strains are neglected, leading to the well-known assumptions in the simple beam theory that plane sections remain plane and normal to the neutral axis after deformation.

However, this simplification may lead to substantial errors in estimating the deflection of thin-walled beam members unless they are long or they are under pure bending moments.

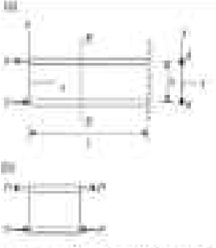


Figure 5.21 Stringer-web beam under pure bending

Consider the thin-walled beam loaded as shown in Figure 5.21a. Note that at the free end no shear stress is applied. We assume that the web can only take shearing stresses, and bending is taken by the two stringers. The bending moment is $M = Ph$ and is uniform over the entire length of the beam.

From the free body diagram in Figure 5.21b, at any section B-B, there is no transverse shear stress (and thus no transverse shear strain) in the web based on the equilibrium condition $\Sigma F_z = 0$.

Hence, we conclude that the axial forces in the stringers also remain constant over the length of the beam. The assumptions adopted by the simple-beam theory are thus valid in this case.

Fig. 5.22: Stringer-web beam subjected to a transverse shear force.

Consider now the beam of Figure 5.21 subjected to a shear load (a constant shear flow q_0) at the free end as shown in Figure 5.22a. From the force and moment balance conditions of the free body shown in Figure 5.22b, we have a constant transverse shear flow $q_z = q_0$ and linearly varying axial force $P = q_0 x h/h = q_0 x$ over the beam length.
The corresponding moment along the beam is $M = Ph = q_0 h x$.

The deflection of the beam consists of two parts; one part results from the bending moment and the other from the transverse shear deformation. The deflection due to bending moment is obtained from the simple beam

$$\frac{d^2 w_0}{dx^2} = -\frac{M_y}{EI_y} = -\frac{q_0 b}{EI_y} x \qquad \boxed{M_y = -EI_y \frac{d^2 w_0}{dx^2}}$$

Integrating twice the above, we obtain

$$w_0 = -\frac{q_0 b}{6EI_y} x^3 + C_1 x + C_2$$

where C_1 and C_2 are arbitrary constants. From the boundary conditions

$$w_0 = 0 \text{ and } \frac{dw_0}{dx} = 0 \text{ at } x = L$$

C_1 and C_2 are determined to be

$$C_1 = \frac{q_0 b L^2}{2EI_y}, \quad C_2 = -\frac{q_0 b L^3}{3EI_y}$$

Thus the deflection at the free end ($x = 0$) due to bending deformation is obtained as

$$w_0^B = -\frac{q_0 b L^3}{3EI_y}$$

The shear deformation in the web contributes to the beam deflection in addition to the bending moment. The constant shear strain in the web is

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{q_0}{tG}$$

in which t is the thickness of the web. Recall that in deriving the simple beam theory, we set $\gamma_{xz} = 0$. Now we relax the condition and allow the transverse shear strain to occur and use the relation

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

To calculate the transverse displacement. At the fixed end ($x = L$), $\partial u_0/\partial z = 0$, which is true over the entire beam length because γ_{xz} is constant. Since γ_{xz} is uniform in the web, we have

$$\gamma_{xz} = \frac{\partial w}{\partial x} = \frac{dw_0}{dx} = \frac{q_0}{tG}$$

along the entire beam. Subsequently, we obtain by integration

$$w_0 = \frac{q_0}{tG}x + C$$

The integration constant C can be determined by the boundary condition $w_0^s = 0$ at $x = L$. The result is

$$C = -\frac{q_0 L}{tG}$$

Thus, the deflection at the free end $(x = 0)$ due to shear deformation in the web is

$$w_0^s = -\frac{q_0 L}{tG}$$

TIMOSHENKO BEAM THEORY

A beam theory that accounts for transverse shear deformation can be developed following the procedure presented in the previous section for the simple beam theory.
The displacement expansions of Eqs. (5.3) and (5.4) are adopted, but the zero transverse shear strain condition Eq. (5.5) is not imposed.
If, again, we assumed that the axial force N is absent, then only transverse deflection w and rotation of cross-section ψ are kept in the formulation.

The only nontrivial strains are:

$$\varepsilon_{xx} = z\frac{d\psi_y}{dx} \qquad \text{[illegible]}$$

$$\gamma_{xz} = \frac{dw_0}{dx} + \psi_y \qquad \text{[illegible]}$$

Since the transverse shear strain γ_{xz} exists, the transverse shear stress τ_{xz} can be calculated directly from γ_{xz} using the stress–strain relations.

The resultant shear force acting on the cross-section can be calculated directly from the shear strain as

$$V_z = \iint_A \tau_{xz}\, dA = \iint_A G\gamma_{xz}\, dA = \iint_A G\left(\frac{dw_0}{dx} + \psi_y\right) dA \qquad \text{[illegible]}$$

$$= GA\left(\frac{dw_0}{dx} + \psi_y\right)$$

where A is the area of the cross-section that carries transverse shear loads.

The bending moment is defined in Eq. (5.8). Using $\sigma_{xx} = E\varepsilon_{xx}$ together with Eq. (5.68) in Eq. (5.8), we have

$\sigma = MZ/I_y$

$E\varepsilon = MZ/I_y$

$M = E\varepsilon I_y / Z$

$$M_y = \iint_A z\sigma_{xx}\, dA \quad (5.8)$$

$$M_y = EI_y \frac{d\psi_y}{dx} \quad (5.71)$$

$$\varepsilon_{xx} = z\frac{d\psi_y}{dx} \quad (5.68)$$

Substituting Eqs. (5.70) and (5.71) in Eqs. (5.18) and (5.17), we obtain

$$EI_y \frac{d^2\psi_y}{dx^2} - GA\left(\frac{dw_0}{dx} + \psi_y\right) = 0 \quad (5.72)$$

$$GA\left(\frac{d^2 w_0}{dx^2} + \frac{d\psi_y}{dx}\right) + p_z = 0 \quad (5.73)$$

$$\frac{dV_z}{dx} = -p_z \quad (5.17) \qquad \frac{dM_y}{dx} = V_z \quad (5.18)$$

These are the two equilibrium equations of the **Timoshenko beam theory**. The two equilibrium equations may be combined into one by eliminating ϕ_z.

This can be performed by first solving Eq. (5.73) for $d\phi_y/dx$ and then substituting it in Eq. (5.72) after differentiating it once. The result is

$$EI_y \frac{d^4 w}{dx^4} = p_z = -\frac{EI_y}{GA}\frac{d^2 w}{dx^2} \qquad (5.74)$$

The boundary conditions at the ends of a Timoshenko beam are specified as follows.

- For a hinged end, $w_0 = 0$, and M_y is prescribed.
- For a clamped end, $w_0 = \phi_y = 0$.
- For a free end, V_z and M_y are prescribed.

Solve the problem of the two-stringer beam shown in Figure 5.22a using the Timoshenko beam theory.

Take advantage of the fact that the shear loading at the free end produces a constant shear force along the beam, i.e.

$$V_2 = q_0 h = GA\left(\frac{dw_3}{dx_1} + \varphi_2\right) = \text{constant} \qquad (5.75)$$

Substitution of Eq. (5.75) in Eq. (5.72) yields

$$\frac{d^2\varphi_2}{dx_1^2} = \frac{q_0 h}{EI_2}$$

$$EI_2\frac{d^2\varphi_2}{dx_1^2} - GA\left(\frac{dw_3}{dx_1} + \varphi_2\right) = 0 \qquad (5.72)$$

and after integration of the differential equation above,

$$\psi_s = \frac{q_0 h}{2EI_s} z^2 + B_1 z + B_2 \qquad \text{([illegible])}$$

where B_1 and B_2 are integration constants.

The boundary conditions are

$M_s = EI_s \frac{d\psi_s}{dz} = 0$ at $z = H$

$V_s = q_0 h = GA\left(\frac{dw_s}{dz} + \psi_s\right)$ at $z = 0$ (this condition has already been used)

$w_s = 0$ at $z = L$

$\psi_s = 0$ at $z = L$

Solving the four equations above with the general solutions:

$$B_1 = 0$$

$$B_2 = -\frac{q_0 h L^2}{EI_s}$$

$$B_3 = \frac{q_0 h L^3}{2EI_s} + \frac{q_0 h}{GA}$$

$$B_4 = \frac{q_0 h L^3}{6EI_s} - B_3 L$$

[illegible]

and thus, the deflection curve:

$$w_0 = -\frac{q_0 b}{6EI_y}\left(x^3 - L^3\right) + B_1(x - L)$$

The maximum deflection occurs at the free end ($x = 0$):

$$w_0(0) = -\frac{q_0 bL^3}{3EI_y} - \frac{q_0 bL}{GA}$$

It is evident that the first term in the solution above is the deflection associated with bending alone and the second term is associated with transverse shear deformation.

In the derivation of the Timoshenko beam theory, two terms in the displacement expansion Eq. (5.1) are kept, resulting in a constant transverse shear strain over the entire cross-section (see Eq. (5.6)).

To compensate for such an oversimplification, it is customary to introduce a shear correction factor k to modify the shear rigidity GA of the cross-section into kGA.

For a rectangular solid cross-section, the shear correction factor is often taken as $k = 5/6$ for static cases and $k = \pi^2/12$ for dynamic cases.

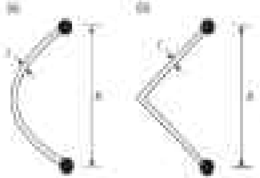


Figure 5.21

Curved and inclined thin-walled sections with stiffeners

Since the Timoshenko beam theory assumes that the resultant shear force acts in the z-direction, only the z-component of the shear flow should be taken into account in calculating the shear force V_z. Using the concept provided by Eq. (5.60), this effect of inclined shear flow can be accounted for by redefining the area A in the shear rigidity GA as the projection of the area of the inclined cross section on the z axis.

For example, for both sections shown in Figure 5.21, $A = th$, rather than the actual cross-sectional area, should be used in the Timoshenko beam equations.

SAINT VENANT'S PRINCIPLE

It is a common practice to adopt the resultant force and resultant moment rather than the actual traction in structural analysis.

An example of a cantilever beam subjected to a shear force F is shown in figure. The actual application of the force could be quite different, namely, it could be the resultant of a distributed shear stress over the cross section at the loading end or the sum of two concentrated forces applied at any two locations on the vertical centerline of the cross section.

We do not have any doubt about the consistency between the solutions obtained utilizing this simplifying approach.

Indeed, such an idealization of loading conditions is justified by **Saint-Venant's principle**.

According to Saint-Venant's principle, the stresses or strains at a point sufficiently far from the locations of two sets of applied loads do not differ significantly if these loads have the same resultant force and moment. These two sets of loads are said to be **statically equivalent**.

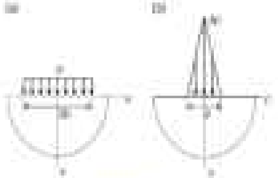


Figure 5.24

Elastic half-space subjected to (a) uniform traction and (b) statically equivalent triangular traction.

An example of two statically equivalent tractions in a two-dimensional problem is given in Figure 5.24. It is obvious that these two traction distributions have the same resultant force and resultant moment. Saint-Venant's principle asserts that the stresses or strains at a distant point in the body produced by these two loads, respectively, do not differ significantly.

In general, the distance at which Saint-Venant's principle works is considered several times the size of the region of load application (the length *h* in this example). In fact, it also depends on the configuration of the body of interest.

SHEAR LAG

Shear lag is a phenomenon of load distribution. Saint-Venant's principle is a good example of shear lag.

In aircraft structures, there are many locations where shear lags occur. The most notable cases are (i) cutouts where some stringers are discontinuous, and (ii) areas of load application.

Cutout in a stringer sheet panel

Figure shows a three stringer-web panel with a cutout. The loads carried by the stringers at the free end are the same. Due to the discontinuity, the load in stringer 5-6 drops to zero at the cutout, while the load in each of the two side stringers increases to 1.5 P in the region. Load redistribution takes place again through shear lag beyond the cutout; the load in stringer 3-4 increases from zero and approaches P if length L is large enough.

The size of the shear lag zone near a cutout depends on the geometry of the structure and material properties of the stringer and the web. If stringers and webs are made of the same material, the shear lag zone is often taken as an approximation to be three times the cutout size *b* (see Figure).

Flexural Shear Flow in Thin Walled Sections

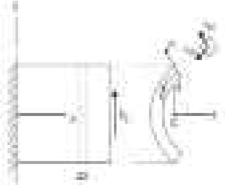
Shear flows in thin-walled sections can be generated by torques or transverse shear forces. The presence of shear strains along the wall gives rise to primary warping (in the longitudinal direction), and end constraints of the structure have significant effects on its structural behavior as discussed in previous chapters.

In this chapter, the shear flow produced by combined torsional and transverse loads is studied.

FLEXURAL SHEAR FLOW IN OPEN THIN-WALLED SECTIONS

Bending stresses in beams with open thin-walled sections subjected to bending loads can be analyzed using the beam equations derived in previous sections with excellent accuracy if the beam span-to-depth ratio is large. In contrast, the transverse shear stresses τ_{xy} and τ_{xz} are very difficult to obtain.

In fact, for a thin-walled section, τ_{xy} and τ_{xz}, in general, are not the most convenient stress components to consider. For instance, it is more advantageous to set up the s-n coordinate system for the thin-walled section shown in the following figure.



Thin-walled section symmetrical with respect to the y-axis

The s axis follows the center line of the wall, and the n axis is perpendicular to the s axis. Referring to the s-n coordinate system, the shear stresses can be represented by τ_{xs} and τ_{xn} as shown in Figure.

Again, using the argument that the wall section is thin and that τ_{xn} vanishes on the boundary, we conclude that τ_{xn} must be small, and we may set $\tau_{xn} = 0$ over the entire wall section as an approximation. Thus, the only nonvanishing shear stress component is τ_{xs} (or simply τ), and a great simplification is achieved.

Symmetric Thin-Walled Sections

If the cross-section is symmetrical about the y or z axis, then $I_{yz} = 0$. For one way bending, say $V_y \neq 0$ and $V_z = 0$, general beam equations are to be used.

Consider the beam section given by Figure, which is symmetrical about the y-axis. We set up the s-n coordinate system as shown in Figure. The shear stress τ_{xs} is assumed to vanish along the longitudinal edges.

Figure 6.2 Free body cut from a beam

Take a free body cut from the beam as shown in Figure 6.2. The balance of forces in the x direction of this free body requires that

$$\iint_{A_s} \Delta\sigma_{xx}\, dA = -q_s\, \Delta x$$

where $q_s = t\tau_{xs}$ is the flexural shear flow on the thin-walled section and A_s is the cross sectional area of the free body.

Taking limit $\Delta x \to 0$, we have

$$\iint_{A_s} \frac{d\sigma_{xx}}{dx}\, dA = -q_s \qquad (6.1)$$

Noting that

$$\sigma_{xx} = \frac{M_z y}{I_z} \quad \text{and} \quad \frac{dM_z}{dx} = V_y$$

we derive the following expression from Eq. (6.1):

$$q_s = -\frac{V_y}{I_z}\iint_{A_s} y\,dA \qquad (6.2)$$

$$= -\frac{V_y Q}{I_z}$$

where

$$Q = \iint_{A_s} y\,dA$$

The shear flow calculated according to Eq. (6.2) is the flexural shear flow because it is induced solely by the bending stress. The resultant force of the shear flow is equal to the applied shear force V_y.

The beam with the channel section shown in Figure 6.3a is loaded with a constant shear force V_y ($V_z = 0$). The wall thickness is [illegible]. Find the flexural shear flow on this section.

Since the section is symmetrical about the y-axis, the flexural shear flow can be calculated using Eq. (6.2).

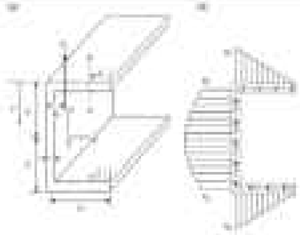


Fig. 8.3 Beam with channel section symmetrical with respect to the z-axis.

Consider an arbitrary section m–m in the upper flange. For this section, we have

$$A_s = t \cdot s_1 \qquad z_c = h$$

Thus,

$$q_1 = -\frac{V_z t h s_1}{I_y}, \qquad 0 \le s_1 \le b$$

This indicates that the shear flow is linearly distributed along the upper flange, and the negative sign means that its direction is opposite that of the coordinate s_1 as shown in Figure 8.3b. The maximum value occurs at $s_1 = b$, i.e.

$$(q_1)_{max} = -\frac{V_z t h b}{I_y} = -q_1$$

In a similar manner, the shear flow on the vertical web is obtained as:

$$q_1 = -\frac{V_y[tbh + ts'(h - s'/2)]}{I_x}$$

where s' is measured starting from the top end of the web. The second term in the numerator in the equation above is the first moment of the cross sectional area of the vertical web up to the point s'.

At the top of the web, $s' = 0$ and

$$q_2 = -\frac{V_y tbh}{I_x} = -q_1$$

At the midsection of the web, $s' = h$, and

$$q_3 = -\frac{V_y th(b + h/2)}{I_x} = -q_1 - \frac{V_y th^2}{2I_x} = -q_3$$

At the bottom of the web, $s' = 2h$, and

$$q_4 = -q_1$$

Again, the negative value of q indicates that the actual direction of q is opposite that of s'.

The shear flow on the lower flange can be calculated using the same counterclockwise shear flow contour.

Alternatively, we may choose a new clockwise contour s'' as shown in Figure 6.3a. We have

$$A_s = ts'', \qquad z_c = -h$$

and

$$q_s = \frac{V_z ths''}{I_y}, \qquad 0 \le s'' \le b$$

The shear flow along the lower flange given by the preceding equation is positive, indicating that its direction is the same as contour s''.

Stringer-Web Sections

For stringer-web constructions such as the one with the cross-section shown in Figure 6.4, stringers are used to take bending. Often, the web can be assumed to be ineffective in bending and its area neglected in the calculation of Q.

As a result, the shear flow between two adjacent stringers becomes constant.

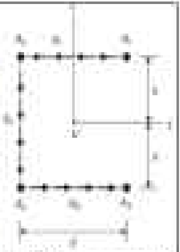


Fig. 6.4 Stringer-web section symmetrical with respect to the y-axis.

Consider a four-stringer thin-walled channel beam with cross-section shown in Figure 6.4. Find the shear flow distribution in the section.

Assume that $A_1 = A_4$ and $A_2 = A_3$. Then the section is symmetric about y axis, and the shear flow Eq. (6.2) can be used.

Thus, shear flow q_i produced by a vertical shear force V_z is obtained as

$$q_i = -\frac{V_z Q_i}{I_y}$$

where

$$Q_i = \sum_{k=1}^{i} z_k A_k$$

$$I_y = 2h^2 (A_1 + A_2)$$

and z_k is the vertical position of stringer A_k.

For q_1, we have $\quad Q_1 = A_1 h$

and thus

$$q_1 = -\frac{V_z A_1 h}{2h^2 (A_1 + A_2)}$$

$$= -\frac{V_z A_1}{2h (A_1 + A_2)}$$

Similarly, we have

$$q_1 = \frac{V_z(A_1 + A_2)h}{2h^2(A_1 + A_2)} = -\frac{V_z}{2h}$$

$$q_2 = \frac{V_z(A_1h + A_2h - A_2h)}{2h^2(A_1 + A_2)}$$

$$= -\frac{V_z A_1}{2h(A_1 + A_2)} = q_3$$

Note that the actual direction of the shear flow is opposite that shown in Figure 6.4.

It is easy to show that the shear flow can also be expressed as

$$q_{i+1} = q_i - \frac{V_z}{I_y} z_{i+1} A_{i+1}$$

where z_{i+1} is the z-coordinate of area A_{i+1}.

The preceding result indicates that

$$V_z = -2hq_1$$

and

$$\sum F_x = hq_1 - hq_3 = 0$$

that is, the vertical resultant of the shear flow must be equal to the applied transverse shear force V_y and the horizontal resultant force must vanish as there is no horizontally applied force.

Unsymmetric Thin Walled Sections

For unsymmetric thin-walled sections under bidirectional bending, the equilibrium Eq. (6.1) is still valid. However, the bending stress σ_x must be calculated from Eq. (5.18).

$$\sigma_{xx} = \frac{I_y M_z - I_{yz} M_y}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y - I_{yz} M_z}{I_y I_z - I_{yz}^2} z$$

$$\sigma_{xx} = (k_y M_z - k_{yz} M_y) y + (k_z M_y - k_{yz} M_z) z \quad (6.3)$$

where

$$k_y = \frac{I_y}{I_y I_z - I_{yz}^2} \qquad k_z = \frac{I_z}{I_y I_z - I_{yz}^2} \qquad k_{yz} = \frac{I_{yz}}{I_y I_z - I_{yz}^2} \quad (6.4)$$

Substituting Eq. (6.3) together with Eqs. (5.25b) and (5.25d) into Eq. (6.1), we obtain

$$q_s = -(k_z V_y - k_{yz} V_z)Q_z - (k_y V_z - k_{yz} V_y)Q_y \quad (6.5)$$

$$\iint_{A_s} \frac{d\sigma_{xx}}{dx} dA = -q_s \quad (6.1)$$

$$\frac{dM_z}{dx} = V_y \qquad \frac{dM_y}{dx} = V_z \quad (5.25)$$

where

$$Q_z = \iint_{A_s} y\, dA, \qquad Q_y = \iint_{A_s} z\, dA$$

are the first moments of the area A_s about the z and y axes, respectively.

Example

Consider the stringer web beam shown in Figure 6.5. Find the shear flow distribution on this section. The shear flow produced by combined vertical load V_y and horizontal load V_x can be solved by considering these two loads separately.

Consider the applied load $V_x = 0$ and $V_y \neq 0$. The direction of the shear flow q is indicated in the figure.

The positive direction of the shear flow is assumed to be the same as that of the contour s.

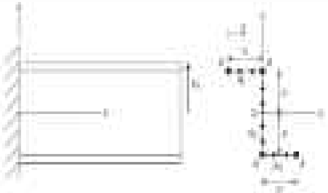


Fig 6.5 Stringer-web beam with an unsymmetrical section.

The positive direction of the shear flow is assumed to be the same as that of the contour s. The moments of inertia and product of inertia of the section are first calculated as

$$I_z = 4Ah^2, \qquad I_y = 2Ah^2, \qquad I_{yz} = -2Ah^2$$

Subsequently, we calculate k_y, k_z and k_{yz} according to Eq. (6.4) with the result

$$k_y = \frac{1}{Ah^2}, \qquad k_z = \frac{1}{2Ah^2}, \qquad k_{yz} = -\frac{1}{2Ah^2}$$

For shear flow q_1, only one stringer is involved. We have

$$Q_y = Ah, \qquad Q_z = -Ah$$

and the shear flow is obtained from Eq. (6.5) as

$$q_1 = -k_{yz}V_zQ_z - k_yV_zQ_y$$
$$= 0$$

For q_2, we have $Q_y = 2Ah, \qquad Q_z = -Ah$

and

$$q_1 = \left(-\frac{1}{2Ah^2}\right)(-Ah)V_y - \left(\frac{1}{2Ah^2}\right)(2Ah)V_y$$

$$= -\frac{1}{2h}V_y$$

In the same manner, we obtain $q_3 = 0$

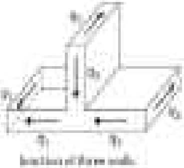
Of course, in this case, constant shear flow q_1 can be obtained from the fact that the resultant transverse shear force must be equal to V_y, i.e.

$$2hq_1 = -V_y$$

[illegible]

Multiple Shear Flow Junctions:

In multicell thin-walled sections, there are junctions where three or more shear flows meet.



Junction of three walls.

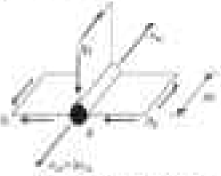
We have shown that for shear flows produced by a torque,

$$q_1 = q_2 - q_3$$

or $q_2 = q_1 + q_3$

This relation is valid for the total shear flows produced by transverse forces for sections without concentrated areas.

For sections consisting of stringers and thin sheets, the relation (5.7) is not valid. Consider the junction of three sheets and a stringer as shown in Fig. 5.?. The equilibrium equation in the x direction is



Junction of three sheets and a stringer

$$\sum F_x = 0: (\sigma_{xx} + \Delta\sigma_{xx})A - \sigma_{xx}A + q_2\Delta x - q_1\Delta x - q_3\Delta x = 0$$

Taking $\Delta x \rightarrow 0$, we have: $q_2 = q_1 + q_3 - A\dfrac{d\sigma_{xx}}{dx}$

For example, consider a symmetrical section subjected to transverse force V_y. We have:

$$\sigma_{xx} = \frac{M_z y_s}{I_z}, \qquad \frac{dM_z}{dx} = V_y$$

where y_s is the vertical position of the stringer. Substitution of Eq. (5.8) into Eq. (5.9) yields

$$q_2 = q_1 + q_3 - \frac{V_y A y_s}{I_z}$$

which is different from Eq. (5.7).

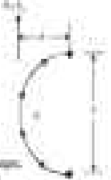
SHEAR CENTER IN OPEN SECTIONS

In calculating the shear flow in an open section produced by shear forces, only the magnitude of the shear force is used. The position of the applied force in the y–z plane is not specified. However, the flexural shear flow resulting from the shear force has a definite resultant force location. This location is called the **shear center** of the cross section.

The shear center is sometimes called the center of twist. If a torque is applied about the shear center, the beam will twist without bending. Conversely, if the shear force is applied through the shear center, the beam will bend without twisting.

To illustrate the procedure of finding the location of shear center, let us consider a thin walled bar with two heavy flanges as shown in figure. Assume that the curved web is ineffective in bending. Consequently, the shear flow is constant between the flanges, and is obtained as

$$q = \frac{V_z}{h} \quad (6.11)$$



Thin-walled bar with two heavy flanges.

The location of the resultant force V_y ($= V_y$) of the shear flow q is also the location of the shear centre. The applied shear force V_y must be applied through this location in order to avoid additional torsional deformation.

Assuming that the shear center is at a distance e to the left of the top flange (see figure 6-8), and requiring that the moment produced by V_y about the top flange be the same as by the shear flow, we have

$$V_y e = 2\bar{A}q = 2\bar{A}\frac{V_y}{h} \qquad (6.12)$$

where

$$\bar{A} = \frac{1}{2}\pi\left(\frac{h}{2}\right)^2 = \frac{1}{8}\pi h^2$$

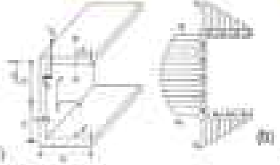
is the area enclosed by the curved web and the straight line connecting the two flanges. From Eq (6.12), we obtain

$$e = \frac{2\bar{A}}{h} = \frac{\pi h}{4}$$

The positive sign of e indicates that the location of the shear center is to the left of the flanges.

In general, the location of a shear center is determined by its horizontal position and vertical position. The horizontal position is obtained from the loading condition $V_x = 0$ and $V_y \neq 0$, and the vertical position is determined using $V_x \neq 0$ and $V_y = 0$.

Find the shear center for the beam with a channel section shown in Figure 6.3. The beam is symmetrical about the x-axis. Thus, $y_s = 0$. To determine the horizontal position of the shear center, we consider the loading case $V_x = 0$, $V_y \neq 0$ for which the shear flow has already been obtained in Example 6.1 and is reproduced in Figure 6.13a.



Assume that V_y passes through the shear center, which is assumed to be at e_x to the right of the vertical wall. Then the moment produced by V_y and the shear flow about any axis (that is parallel to the z-axis) must be equal. By selecting the axis location at the **lower left corner** of the channel, the shear flows on the vertical wall and the lower flange produce no moment and only the shear flow on the upper flange does.

The shear flow on the upper flange can be written in the form

$$q = \frac{s}{b} q_1$$

The moment of this shear flow about the axis selected is clockwise and is given by

$$\int_0^b 2hq\,ds = \frac{hb^2 q_1}{b} = hbq_1$$

For the assumed loading position (see figure 6.11), the moment produced by V_y about the same axis is $V_y e_x$, which is counterclockwise. Hence,

$$V_y e_x = -hbq_1$$

Noting that

$$q_1 = \frac{V_y d h}{I_y}$$

we obtain

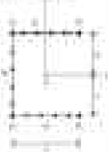
$$\xi_s = -\frac{dt^2 h^2}{I_y}$$

The minus sign indicates that the actual location of the shear center is to the left of the vertical web. Thus, a fitting sticking out from the vertical web may be necessary to facilitate such loading.

Example 9.4

The four-stringer thin-walled channel section of Figure 6.18 is a special case of the section given in Figure 6.4. Find the horizontal and vertical positions of its shear center.

To determine the horizontal position of its shear center we consider the loading of $V_x = 0$ and $V_y = 0$. The resulting shear flow has been obtained in Example 6.2. We have:



$$q_1 = \frac{V_y A_1}{2h(A_1 + A_2)}$$

$$q_2 = \frac{V_y}{2h}$$

$$q_3 = \frac{V_y A_1}{2h(A_1 + A_2)}$$

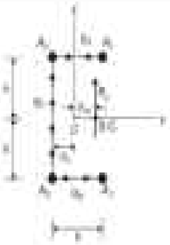


Fig 6.50 Shear flow direction

Note that the shear flow direction in Figure 6.50 is the actual direction. Let the resultant force of the shear flow be R, which is obviously equal to V in magnitude and should pass through the shear center (see Figure 6.51).

The resultant force R must also produce the same moment about the x axis (or any other axis) as does the shear flow, i.e.

$$R_1 y_o = -q_1(h)(h) - q_2(2h)(\bar{y}_c) - q_1(h)(h) \quad (6.1)$$

where $\bar{y} = \dfrac{A_2 b}{A_1 + A_2}$

is the horizontal distance between the centroid and the vertical web.

Using $B_1 = B_3$ and the expressions for q_1, q_2, and q_3, we obtain the distance y_s from Eq. (6.17) as

$$y_s = -\frac{2M_1}{A_1 + A_2}$$

The location $y = y_s$ is the horizontal position of the **shear center** for this section for the shear force applied in the vertical (z) direction. The negative sign of y_s in Eq. (6.20) indicates that the actual shear center is to the left of the centroid.

If the shear force V_z is applied through the shear center, this shear flow is the complete response of the structure. If the shear force V_z is not applied through the shear center as shown in Figure 6.11a, then it results in an additional torque load. As shown in Figure 6.11b, the shear force V_z can be translated to the shear center, resulting in a torque $V_z d$. In such cases, the shear stresses produced by the torque must be added to the flexural shear stresses.

Since open thin-walled sections are generally weak in torsion, it is desirable to apply the shear force through the shear center.

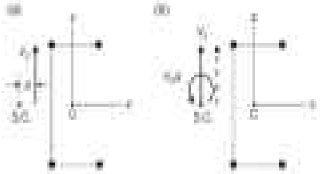


Fig. 6.11 Location of shear force: (a) actual location; (b) shifted to shear center.

To find the vertical position e_z of the shear center, we consider the section of Figure 6.10 subjected to loading $V_z = 0$ and $V_y \neq 0$. For this section $I_{yz} = 0$, and from Eq. (6.C) we have

Fig. 6.12 Shear flow for a horizontal shear force.

$$q_s = -(k_x V_x - k_{xy} V_y) Q_y - (k_y V_y - k_{xy} V_x) Q_x \quad (6.5)$$

$I_y \neq 0$, $V_x = 0$ and $k_{xy} = 0$

$$q_s = -k_y V_y Q_x \quad (6.17)$$

The location of the centroid is indicated in Figure 6.10 with

$$\bar{y} = \frac{A_1 h}{(A_1 + A_2)}$$

The shear flow is easily obtained from Eq. (6.17):

$$q_1 = -k_y V_y A_1 (h - \bar{y}) = -k_y V_y \frac{A_1 A_2 h}{A_1 + A_2} = -q_0$$

$$q_2 = q_1 - k_y V_y A_2 (-\bar{y}) = 0$$

$$q_3 = q_2 - k_y V_y A_3 (-\bar{y}) = k_y V_y \frac{A_1 A_2 h}{A_1 + A_2} = -q_1 = q_0$$

The shear flow after adjusting the sign for direction, is shown in Figure 6.11. The shear flow is seen to be symmetrical with respect to the y-axis. The resultant V_y thus coincides with the y-axis; consequently, the vertical location of the shear center is $x_0 = 0$.

Simple Rule for Determining the Shear Center

The following rule can be used to locate the shear center for sections possessing symmetries. If a section (including both stringers and thin webs) is symmetric about an axis, then the shear center lies on this axis.

For example, the sections of Figure 6.13a, b are symmetrical with respect to the y axis and thus $z_s = 0$.

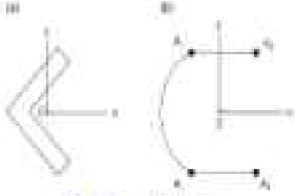


Fig 6.13 Symmetrical sections

This cannot be said about the section of Figure 6.14. Although it is considered a symmetrical section for bending when the thin webs are assumed ineffective in bending, the shear flow cannot be symmetric about the y-axis because of the unsymmetric webs.

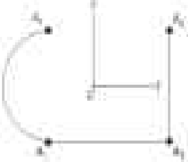


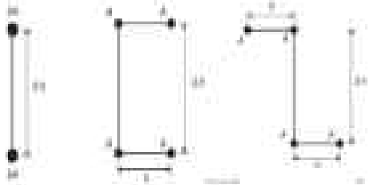
Fig. 6.14 Symmetric section.

For sections such as the angle section in Figure 6.11a, the T-section in Figure 4.28a, and cruciform sections in which all walls meet at a single location, the shear center coincides with the intersection point. This conclusion is easily reached in view of the fact that all shear flows are straight and meet at one point.

Figure 4.28a

Figure 6.11a

The stringer-web sections shown in Figures 5.23–5.25 are subjected to the shear force $V_y \neq 0$, while $V_x = 0$. Find the bending stresses in the stringers for the same bending moment M_x. Which section is most effective in bending?



(4) Figure 4.20

(1) Because of symmetry, the centroid is located at the middle of the vertical web.

(2) Moment of inertia

$$I_y = \sum_i A_i z_i^2 = 2(2A \times h^2) = 4Ah^2$$

$$I_z = \sum_i A_i y_i^2 = 2(2A \times 0^2) = 0$$

$$I_{yz} = \sum_i A_i y_i z_i = 0$$

(3) Bending stress. Considering $M_y \neq 0$ and $M_z = 0$

$$\sigma_{xx} = \frac{M_y}{I_y} z = \frac{M_y}{4Ah^2} z$$

The stresses at the stringers are

① At $z = h$: $\sigma_{xx} = \frac{M_y}{4Ah^2} h = \frac{M_y}{4Ah}$

② At $z = -h$: $\sigma_{xx} = -\frac{M_y}{4Ah^2} h = -\frac{M_y}{4Ah}$

(b) Figure 4.21

(1) Because of symmetry (when neglecting the effects of webs), the centroid is located at the center of the section as shown in the figure.

(2) Moment of inertia

$$I_y = \sum_{1}^{4} A_i z_i^2 = 4(A \times h^2) = 4Ah^2$$

$$I_z = \sum_{1}^{4} A_i y_i^2 = 4(A \times (\frac{b}{2})^2) = Ab^2$$

$$I_{yz} = \sum_{1}^{4} A_i y_i z_i = 0$$

(3) Bending stress: Considering $M_y \neq 0$ and $M_z = 0$

$$\sigma_{xx} = \frac{-I_{yz} M_y}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y}{I_y I_z - I_{yz}^2} z = \frac{M_y}{I_y} z$$

The stresses at the stringers are (position is not involved)

① At $z = h$, $\sigma_{xx} = \frac{M_y}{4Ah^2} z = \frac{M_y}{4Ah}$

② At $z = -h$, $\sigma_{xx} = \frac{M_y}{4Ah^2} z = -\frac{M_y}{4Ah}$

(d) Figure 4.22

(1) Again, when neglecting the effects of webs, the centroid is located at the middle of the vertical web.

(2) Moment of inertia

$$I_y = \sum_i A_i z_i^2 = 4(A \times h^2) = 4Ah^2$$

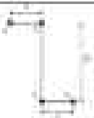
$$I_z = \sum_i A_i y_i^2 = 2(A \times h^2) = 2Ah^2$$

$$I_{yz} = \sum_i A_i y_i z_i = 2(A \times h \times (-h)) = -2Ah^2$$

(3) Bending stress. Considering $M_y \neq 0$ and $M_z = 0$

$$\sigma_x = \frac{-I_{yz} M_y}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y}{I_y I_z - I_{yz}^2} z = \frac{2M_y}{(4 \times 2 - (-2)^2) A h^2} y + \frac{2M_y}{(4 \times 2 - (-2)^2) A h^2} z$$

$$= \frac{M_y}{2Ah^2} y + \frac{M_y}{2Ah^2} z$$



The stresses of the stringer are

At $z = h$, $y = -h$,

$$\sigma_{xx} = \frac{M_z}{2th^3} y + \frac{M_y}{2th^3} z = \frac{M_y}{2th^3}(-h+h) = 0$$

① At $z = h$, $y = 0$,

$$\sigma_{xx} = \frac{M_z}{2th^3} y + \frac{M_y}{2th^3} z = \frac{M_y}{2th^3}(0+h) = \frac{M_y}{2th}$$

② At $z = -h$, $y = 0$,

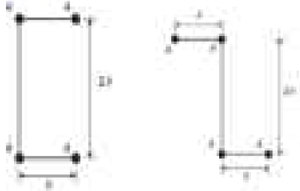
$$\sigma_{xx} = \frac{M_z}{2th^3} y + \frac{M_y}{2th^3} z = \frac{M_y}{2th^3}(0-h) = -\frac{M_y}{2th}$$

③ At $z = -h$, $y = h$,

$$\sigma_{xx} = \frac{M_z}{2th^3} y + \frac{M_y}{2th^3} z = \frac{M_y}{2th^3}(h-h) = 0$$

(c) Comparing the above results, sections in Figure 4.20 and Figure 4.21 are both more effective than the section in Figure 4.22 for this particular loading.

Compare the bending capabilities of the two sections of Figures 4.21 and 4.22 ($M_y = 0$, $M_z \neq 0$).



(a) Figure 4.21

(1) The centroid is located at the center of of the space defined by the four stringers.

(2) Moment of inertia:

$$I_y = \sum A z_i^2 = 4(A \times b^2) = 4Ab^2$$

$$I_z = \sum A y_i^2 = 4(A \times (\frac{b}{2})^2) = Ab^2$$

$$I_{yz} = \sum A y_i z_i = 0$$

(3) Bending stress

Considering $M_y = 0$ and $M_z \neq 0$ we have

$$\sigma_{xx} = \frac{I_y M_z}{I_y I_z - I_{yz}^2} y + \frac{-I_{yz} M_z}{I_y I_z - I_{yz}^2} z = \frac{M_z}{I_z} y$$

The stresses in stringers 1 and 4 are:

$$(A)\ \ y = \frac{h}{2},\ \ \sigma_{xx} = \frac{M_z}{Ah^2} y$$

The stresses in stringers 2 and 3 are:

$$(B)\ \ y = -\frac{h}{2},\ \ \sigma_{xx} = \frac{M_z}{Ah^2} y$$

(b) Figure 4.22

(1) The centroid is located at the middle of the vertical web.

(2) Moment of inertia

$$I_y = \sum_i A_i z_i^2 = 4(Ah^2) = 4Ah^2$$

$$I_z = \sum_i A_i y_i^2 = 2(Ah^2) = 2Ah^2$$

$$I_{yz} = \sum_i A_i y_i z_i = 2(A h h + (-h)h) = -2Ah^2$$

(3) Bending stress

For $M_y = 0$ and $M_z \neq 0$

$$\sigma_{xx} = \frac{I_y M_z}{I_y I_z - I_{yz}^2} y + \frac{-I_{yz} M_z}{I_y I_z - I_{yz}^2} z = \frac{4M_z}{[(4/3) - (-1)^2] t h^3} y + \frac{2M_z}{[(4/3) - (-1)^2] t h^3} z$$

$$= \frac{M_z}{th^3} y + \frac{M_z}{2th^3} z$$

The stress at point 1 is

At $z = h$, $y = -h$

$$\sigma_{xx} = \frac{M_z}{th^3} y + \frac{M_z}{2th^3} z = \frac{M_z}{2th^3}(-2h + h) = -\frac{M_z}{2th^2}$$

Point 2

At $z = h$, $y = h$: $\sigma_{xx} = \frac{M_z}{th^3} y + \frac{M_z}{2th^3} z = \frac{M_z}{2th^3}(h + h) = \frac{M_z}{2th^2}$

Point 3

At $z = -h$, $y = h$: $\sigma_{xx} = \frac{M_z}{th^3} y + \frac{M_z}{2th^3} z = \frac{M_z}{2th^3}(h - h) = -\frac{M_z}{2th^2}$

Point 4

At $z = -h$, $y = h$: $\sigma_{xx} = -\frac{M_z}{th^3} y + \frac{M_z}{2th^3} z = -\frac{M_z}{2th^3}(2h - h) = \frac{M_z}{2th^2}$

Comparing the above results, we see that sections in Figure 4.21 and Figure 4.22 have the same bending efficiency: they both reach the same maximum bending stress under the same moment.

Find the flexural shear flow produced by the transverse shear force V_z = [illegible] in the beam with the thin-walled section given by Fig. [illegible]

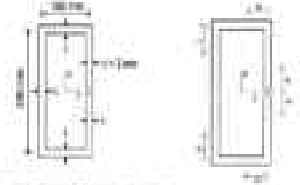


Figure [illegible] Thin-walled section with a slit cut

Solution:

(a) Assume that the transverse shear force acts through the shear centre, and, thus, no torsional effect exists. Assume that the loss of material at the cut is negligible. Hence the centroid of the cross-sectional area is obviously at the centre as shown in Fig. [illegible]

The shear flow is obtained by

$$q_s = -\frac{V_z Q}{I_y} \quad \text{[illegible]}$$

where $Q = \int z\, t\, ds = A_s \bar{z}$ is the first moment of area A_s (the area concerned)

$$I_y = \frac{1}{12}[\text{illegible}]$$

(1) [illegible]

$Q = [illegible]$

$q_1 = -\frac{V_y Q}{I_x} = [illegible]$

At [illegible]

At [illegible] (The negative sign means that the actual direction of the shear flow is opposite to assumed direction of s_1)

[illegible]

(2) [illegible]

The first moment Q for the segment [illegible] Thus

$Q = [illegible]$

$q_2 = -\frac{V_y Q}{I_x} = [illegible]$

Note that the distribution of the shear flow is linear along the [illegible]

At [illegible]

At [illegible]

[illegible]

(3) On [illegible]

[illegible]

[illegible]

$q = $ [illegible]

[illegible]

At [illegible]

At [illegible]

(4) On [illegible]

For convenience, [illegible]

$Q = $ [illegible]

$q = -\frac{V_y Q}{I_z} = $ [illegible]

At [illegible]

At [illegible]

(3) On x_2: 0 ~ 0.1m

$$Q=-4000(0.1)^2-4000x_2(0.1)=-10^{[illegible]}-[illegible]\times10^{[illegible]}x_2$$

$$q_2=-\frac{VQ}{I_z t}=\frac{1000\times(10^{[illegible]}+2\times10^{[illegible]}x_2)}{6.6675\times10^{[illegible]}}=1500+3\times10^4x_2$$

$$x_2=0m,\quad q_2=1500N/m$$

$$x_2=0.1m,\quad q_2=1500+3\times10^4\times0.1=4500N/m$$

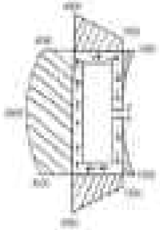
(4) On x_3: 0 ~ 0.1m

$$Q=-4000(0.1)^2-4000(0.1)^2-4000x_3(0.1-0.5x_3)$$
$$=-8\times10^{[illegible]}-2\times10^{[illegible]}(x_3-5x_3^2)$$

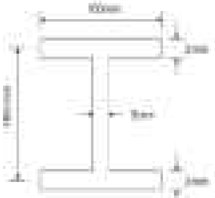
$$q_3=-\frac{VQ}{I_z t}=\frac{1000\times[8\times10^{[illegible]}+2\times10^{[illegible]}(x_3-5x_3^2)]}{6.6675\times10^{[illegible]}}$$
$$=4500+30000x_3-150000x_3^2$$

$$x_3=0m,\quad q_3=4500+30000(0)-150000(0)^2=4500N/m$$

$$x_3=0.1m,\quad q_3=4500+30000(0.1)-150000(0.1)^2=6000N/m$$



Find the shear flow of the wide-flange beam [Figure 6.22] subject to $V_y = 1000$ N.



Solution.

(a) Assume that the transverse shear force acts through the shear center and positions [illegible]. From symmetry, it is obvious that the centroid of the section is located at the mid point of the vertical web.

The shear flow is obtained by

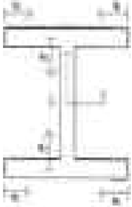
$$q_s = -\frac{V_y Q_x}{I_x} \qquad (7.3.1)$$

where $Q_x = \iint y \, dA$ [illegible] is the first moment of area [illegible] and [illegible] is the vertical distance from the centroid of [illegible] to the [illegible] axis.

The moment of inertia of the cross-section is

$$I_x = \frac{1}{12}[\text{illegible}]$$

(b) Set up the shear flow contours as in the following figure.



(1) On s_1: 0 – 0.05m

$$Q=\iint_{A_1} y\,dA = A_1\bar{y}_1 = (s_1)\left(\frac{0}{2}\right)+(0.003s_1)\left(\frac{0.1}{2}\right)=1.5\times10^{-4}s_1$$

$$q_1=-\frac{V\,Q}{I_z}=-\frac{1000\times1.5\times10^{-4}s_1}{1.7298\times10^{-6}}=-8.675\times10^{4}s_1$$

At $s_1=0$, $q_1=-8.675\times10^{4}\times0=0$

At $s_1=0.05\text{m}$, $q_1=-8.675\times10^{4}\times0.05=-4338\,N/m$

(2) On s_2: 0 – 0.05m

This is similar to s_1

$$q_2=-8.675\times10^{4}s_2$$

At $s_2=0$, $q_2=0$

At $s_2=0.05\text{m}$, $q_2=-8.675\times10^{4}\times0.05=-4338\,N/m$

(3) On $x_3(t) = 0.05m$

[illegible]

(4) On $x_4(t) = -0.05m$ & $x_5(t) = 0.05m$

The equation is the same as that of x_2 and x_1 with opposite sign.

(5) On $x_6(t) = 0.05m$

The equation is the same as that of x_3 with opposite sign.

Find the flexural shear flow in the section of Figure 6.13 for $S_y = 5000$ N.

Solution:

(a) Assume that the transverse shear force acts through the shear center and produces no torsion to the cross-section. Also assume that the thin sheets are ineffective in bending. Let the left-bottom stringer be the origin of a coordinate system with respect to which we now determine the centroid of this four-stringer section. We obtain

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{[illegible]}{[illegible]} = [illegible]$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{[illegible]}{[illegible]} = [illegible]$$

Now we set up the (x, y) coordinate system with the origin located at the centroid. The moments of inertia with respect to y-axis and x-axis are obtained as

$$I_y = \sum A_i x_i^2 = [illegible]$$

$$I_x = \sum A_i y_i^2 = [illegible]$$

(b) The shear flow calculation

The shear flow formula for symmetric and open sections

$$q_s = -\frac{V_y Q_x}{I_x}$$

Is used with the contour direction shown in Fig 5.32.

(1) On s_1

$$Q_x = (2A_b)(0.5h) = A_b h$$

$$q_1 = -\frac{V_y Q_x}{I_x} = -\frac{5000 \times A_b h}{1.5 A_b h^2} = -\frac{3333.33}{h}$$

(2) On s_2

$$Q_x = (3A_b)(0.5h) = 1.5 A_b h$$

$$q_2 = -\frac{V_y Q_x}{I_x} = -\frac{5000 \times 1.5 A_b h}{1.5 A_b h^2} = -\frac{5000}{h}$$

(3) On s_3

$$Q_x = (3A_b)(0.5h) - (A_b)(0.5h) = A_b h$$

$$q_3 = -\frac{V_y Q_x}{I_x} = -\frac{5000 \times A_b h}{1.5 A_b h^2} = -\frac{3333.33}{h}$$

The negative signs indicates that the actual shear flow direction is opposite to the assumed direction of the contour.

CLOSED THIN-WALLED SECTIONS AND COMBINED FLEXURAL AND TORSIONAL SHEAR FLOW

Closed thin-walled sections are capable of taking both shear forces and torques. Shear flows can result from simultaneous applications of shear forces and torques.

In the derivation of flexural shear flows in open sections, the flexural shear stress τ_{xs} (and thus, q_s) is zero at the free edges (see Figure 6.16).

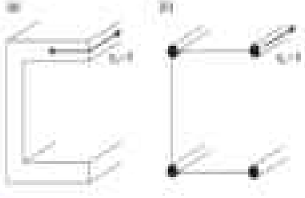


Fig. 6.16 Flexural shear flows in open sections

For closed sections, such as shown in Figure 6.17a, there are no free edges. We assume that at point O the value of the shear flow is q_0 (see Figure 6.17b).

Fig. 6.17 Flexural shear flow in a closed section: (a) closed section; (b) section with a fictitious cut.

Thus, the closed section can be regarded as an open section with a nonzero shear flow at point O. Starting contour s from this point (see Figure 6.17b), we obtain the shear flow q_s as

$$q_s = q_s' + q_0 \qquad (6.18)$$

Where q_s' is the shear flow calculated assuming a free edge at point O. Hence the actual shear flow can be considered as the superposition of q_s' and the unknown constant shear flow q_0 as depicted in Figure 6.18.

$$q_s = q_s' + q_0$$

Fig. 6.18 Superposition of shear flows.

The fictional shear flow q' can be regarded as the shear flow produced by the shear force in the open section obtained by cutting the wall longitudinally at point O.

Taking moment about the z axis, we have

$$V_y \cdot d = 2\bar{A}q_0 + \text{moment produced by } q_s' \text{ about the z-axis} \quad (6.19)$$

where $\bar{A}$ is the area enclosed by the shear flow.

Equation (6.19) ensures that the resulting shear flow must produce the same moment as the applied shear force V_y. This equation is used to determine q_0.

Shear Center

The shear flow given by Eq. (6.18) may contain flexural shear and torsional shear if V_z is applied at an arbitrary location.

If the applied shear force V_z passes through the shear center, i.e., $d = y_s$, then the resulting shear flow is pure flexural shear, which should produce no twist angle, i.e.,

$$\theta = 0 = \frac{1}{2GA}\oint \frac{q}{t}\,ds \qquad (6.20)$$

Equation (6.20) is used to determine q_0 in terms of y_s. The location (y_s) of the shear center, if not given, is subsequently obtained from solving Eq. (6.18) by replacing d with y_s.

An equivalent problem to that of Figure 6.17 can be obtained by translating the shear force V_z from $y = d$ to $y = y_s$ (the shear center) and adding a torque $T = V_z(d - y_s)$ as shown in Figure 6.18. The shear flow resulting from this torque must be added to the shear flow produced by the shear force that passes through the shear center.

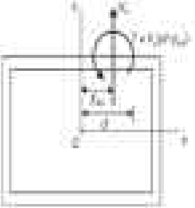


Fig. 6.19 Added torque due to shifting of the shear force.

Example

A four-stringer box beam is loaded as shown in Figure 6.20. Find the shear center.

Assume the skin shown to be ineffective in bending. The centroid is easily calculated and is shown in the figure.

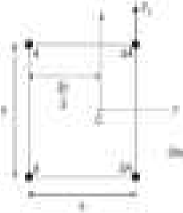


Fig. 6.20 Four stringer box beam.

As far as bending is concerned, this cross-section is symmetrical with respect to the y-axis. Thus, $I_{yz} = 0$. The other properties of the cross section are given by

$$I_y = \frac{1}{3}Ah^2$$

$$I_z = \frac{4}{3}Ah^2$$

We first calculate the shear flow q' by assuming a cut (see Figure 6.21) in the wall between stringers 1 and 2, i.e., $q'_{21} = 0$. The shear flows on other walls are calculated according to Eq. (6.2) for symmetrical sections. We obtain

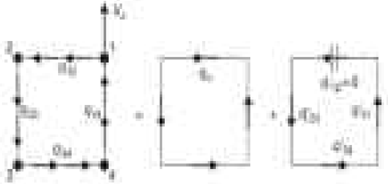


Fig. 6.21 Superposition of shear flows.

$$q'_{23} = -\frac{V_z \cdot h \cdot h/2}{\frac{3}{2}h^3} = -\frac{V_z}{3h}$$

$$q'_{12} = 0$$

$$q'_{31} = -\frac{V_z \cdot 2h \cdot (-h/2)}{\frac{3}{2}h^3} = \frac{2V_z}{3h}$$

The resulting moment of the total shear flow $q = q' + q_0$ must be equal to the moment produced by V_z. Taking moment about stringer 1 and using Equation (6.19) we obtain

$$V_z \cdot 0 = q'_{23} \cdot h \cdot h + 2A q_0$$
$$= q'_{23} h^2 + 2h^2 q_0 \qquad (6.21)$$

Thus,

$$q_0 = -\frac{1}{2} q'_{23} = \frac{V_z}{6h}$$

The total shear flows are

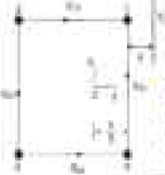
$$q_{12} = q'_{12} + q_0 = \frac{V_y}{6h}$$

$$q_{23} = q'_{23} + q_0 = -\frac{V_y}{6h}$$

$$q_{34} = q'_{34} + q_0 = \frac{V_y}{6h}$$

$$q_{41} = q'_{41} + q_0 = \frac{5V_y}{6h}$$

To determine the horizontal location of the shear center, we assume that the shear force V_y is applied through the shear center (assumed to be at (0,22)) distance e from stringer 1 as shown in Figure 6.22. Then the moment Eq. (6.21) is replaceable by



$$V_y \cdot e = q'_{23} h^2 + 2h^2 q_0$$

Fig. 6.22 Assuming shear force passes through the shear center

Thus,

$$q_0 = \frac{V_y e}{2h^2} - \frac{q_{12}}{2} = \frac{V_y}{6h^2}(h + 3e)$$

and

$$q_{12} = q_0 = \frac{V_y}{6h^2}(h + 3e)$$

$$q_{23} = \frac{V_y}{6h^2}(-h + 3e)$$

$$q_{34} = \frac{V_y}{6h^2}(h - 3e)$$

$$q_{41} = \frac{V_y}{6h^2}(3h + 3e)$$

Since V_y passes through the shear center, the twist angle is equal to zero.
Using Eq. (6.28), we have

$$\theta = \frac{1}{2GA}\left[q_{12}\cdot\frac{h}{t} + q_{23}\cdot\frac{h}{t} + q_{34}\cdot\frac{h}{t} + q_{41}\cdot\frac{h}{t}\right] = 0$$

This equation reduces to

$$q_{12} + q_{23} + q_{34} + q_{41} = 0$$

Solving the above equation for e, we obtain:

$$e = -\frac{1}{2}h$$

The negative sign indicates that the shear center is located to the left of the vertical wall between stringers 1 and 2. It is obvious from Figure 6.22 that

$$y_{sc} = \frac{h}{3} + e = -\frac{h}{6}$$

The vertical location z_{sc} of the shear center can be determined in a similar manner by applying a horizontal shear force V_y. The result is $z_{sc} = 0$, i.e. the shear center lies on the axis of symmetry of the cross-section.

Statically Determinate Shear Flow

At any cross-section of a thin-walled beam, the shear flow must result in the same resultant force and moment as the applied ones, i.e.

$$\sum F_x = V_x \qquad (6.23a)$$

$$\sum F_y = V_y \qquad (6.23b)$$

$$\sum M = V_x e_y + V_y e_x \qquad (6.23c)$$

where e_x and e_y are the distances of V_y and V_x from the axis about which the moments are taken.

For some sections, the shear flow can be determined from these equations alone. This type of shear flow is **statically determinate**. In this case, the sectional properties (I_x, I_y and I_{xy}) are not involved.

Example 6.7

Consider a three-stringer single-cell section loaded as shown in Figure 6.23. Obtain the shear flow and vertical location of the shear center.

Fig. 6.23 Three-stringer single-cell section.

The three equations on the equivalent resultants are given by

$$\sum F_z = 0; \quad bq_1 - bq_2 = 0 \qquad \text{(a)}$$

$$\sum F_y = V_y; \quad 2bq_3 - bq_1 - bq_2 = V_y \qquad \text{(b)}$$

$$\sum M_x = V_y e_z; \quad \pi b^2 q_1 = V_y e_z \quad [q_2 = q_1 \text{ from (a) is used}] \qquad \text{(c)}$$

Solving the equations above, we obtain:

$$q_1 = q_2 = \frac{V_y e_x}{\pi h^2}$$

$$q_3 = \frac{(\pi h + 2e_x)V_y}{2\pi h^2}$$

If V_y passes through the shear center, then the twist angle $\theta =$ 0, i.e.

$$\frac{q_1 \cdot \pi h/2}{t} + \frac{q_2 \cdot \pi h/2}{t} + \frac{q_3(2h)}{t} = 0$$

This leads to:

$$e_x = -\frac{\pi h}{2+\pi}$$

The negative sign indicates that the shear center is located to the left of the vertical axis.

Example

The cross section of a three-stringer thin-walled beam and the applied loads are shown in Figure 6.24a. Find the shear flow.

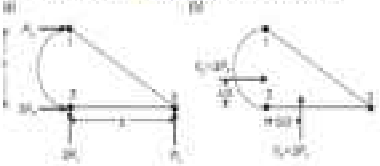


Fig. 6.24 Resultant shear forces.

This problem can be solved by first converting the loads into the resultant shear forces as shown in Figure 6.24b and then using the method of cutting the closed cell into an open cell to find q_0.

Alternatively, by recognizing that the shear flow is statically determinate, we can use Eq. (6.23) to determine the shear flow.

Assuming constant shear flows in the thin wall segments as shown in Figure 6.25, we have

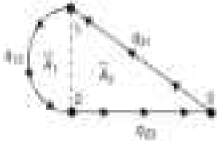


Fig. 6.25 Assumed shear flows.

$$3P_x = hq_{23} - hq_{31} \qquad [6.27]$$

$$3P_y = -hq_{12} + hq_{31} \qquad [6.28]$$

$$2P_y h + P_x b = 2\bar{A}_1 q_{12} + 2\bar{A}_2 q_{23} \qquad [6.29]$$

In deriving the equations above, the relation given by Eq. (6.32) has been used. Also, the areas $\bar{A}_1$ and $\bar{A}_2$ are The

$$\bar{A}_1 = \frac{1}{2}\pi\left(\frac{h}{2}\right)^2 = \frac{1}{8}\pi h^2$$

$$\bar{A}_2 = \frac{1}{2}hb$$

The shear flows q_{12}, q_{23}, and q_{34} are obtained by solving the three Eqs (6.27)–(6.29).

Example

Find the unknown shear flow in the section shown in Figure 6.26 for V_y = 10 kN. Given that h = 300 cm, area A of each stringer = 10 cm², and b = 0.8h.

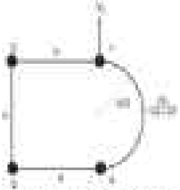


Fig. 6.26 Four-stringer single-cell closed section

This is a closed single-cell section. As such, the problem can be solved using the method shown in Example 6.7. As the shear flow, since q_{12} is already provided, there is no need to make a cut and start solving the problem as an open section.

The shear flow on the remaining skins can be obtained directly as

$$q_{31} = -\frac{V_y A_1^2}{I_x} + q_{s0} = -\frac{V_y}{2h} + \frac{3V_y}{4(2+\pi)h} = -5000 + 1458 = -3542\ \mathrm{N/m}$$

$$q_{41} = -\frac{V_y}{2h} - \frac{V_y}{2h} + \frac{3V_y}{4(2+\pi)h} = -10000 + 1458 = -8542\ \frac{\mathrm{N}}{\mathrm{m}}$$

$$q_{23} = \frac{V_y}{2h} - \frac{V_y}{h} + \frac{3V_y}{4(2+\pi)h} = -5000 + 1458 = -3542\ \frac{\mathrm{N}}{\mathrm{m}}$$

Recall that shear flow on any skin depends on the shear flow on the preceding skin. When a section is open, then the shear flow is zero at the free end. On a close section, shear flow on any section is not known. For such a section, Eq (6.2) cannot be directly applied, and a cut is usually necessary. In this case, the shear flow q_o is known.

$$q_s = -\frac{V_y}{I_x}\int_0^s y\,t\,ds = -\frac{V_y Q}{I_x} \qquad (6.2)$$

CLOSED MULTICELL SECTIONS

As discussed in the previous section, the shear flow in a single-cell beam can be analyzed by making a fictitious cut so that it can be treated as an open section with an existing constant shear flow q_o. The shear flow q' is unambiguously obtained from Eq. for the "open section" subjected to the applied shear forces.

The unknown shear flow q_o is determined from the requirement that the moment produced by the total shear flow $q' + q_o$ must be equal to the moment produced by the applied shear forces.

The aforementioned procedure can be employed for the analysis of shear flows in beams with multicell thin-walled cross-sections.

Furthermore, consider an n-cell section. Make a "cut" in the wall in each cell to make the entire section "open."

For each cell, a constant shear flow q_i ($i = 1, 2, \ldots, n$) must be added to the shear flow q' calculated for the open section.

It requires n equations to solve for the n unknowns q_i.

These n equations are provided by the n − 1 compatibility equations,

$$\theta_1 = \theta_2 = \cdots = \theta_n$$

where θ_i is the twist angle per unit length of the i^{th} cell.

An additional equation is provided by equating the moment of the applied shear forces to the total resultant moment of all the shear flows in the cells.

In making the cuts, no part of the section should be completely cut off. In setting up the shear flow contours for the resulting open section, it is more convenient to begin each contour from the cut location where $q' = 0$.

Moreover, each wall can be covered by only a single contour. Figure 6.27 shows a few possible ways to cut a three-cell box beam section. Apparently, the cut depicted in Figure 6.27c is the most convenient because a single contour is sufficient.

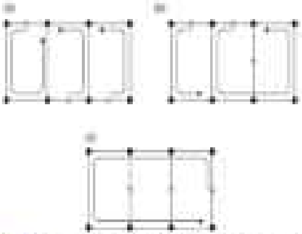


Fig. 6.27 A few possible ways to cut a three-cell box beam section.

Example 6.18

The two-cell box beam section shown in Figure 6.23 is symmetrical about the y-axis. Find the shear flow and shear center.

Fig. 6.23 Geometry and shear flow contours

Assume that the skins are ineffective in bending. The pertinent cross sectional property is

$$I_x = 2(1 + 2 + 3)(20)^2 = 4800 \text{ cm}^4$$

Cut the skin between stringers 1 and 2 and between 5 and 6, and set up the contours 1 and 2 as shown in Figure 6.23. Each contour must start from the cut (the free edge). The positive shear flow direction is assumed to be in the contour direction. In the following, q_{ij} is used to denote the shear flow between stringers i and j.

Cell 1:

$$q_{12} = 0$$

$$q_{23} = -\frac{4000}{4000}(1\times 20) = -20\ \text{N/mm}$$

$$q_{31} = q_{23} - \frac{4000}{4000}(1\times -20) = 0 \qquad (6.21)$$

Cell 2:

$$q_{14} = 0$$

$$q_{45} = -\frac{4000}{4000}(2\times 20) = -40\ \text{N/mm}$$

$$q_{56} = q_{45} + q_{23} - \frac{4000}{4000}(1\times -20) = 0 \qquad (6.22)$$

$$q_{61} = q_{56} - \frac{4000}{4000}(1\times -20) = 40\ \text{N/mm}$$

The shear flows are completed by adding the constant shear flows q_1 and q_2 (see Figure 6.29) in the individual cells, respectively. The equations needed for determining q_1 and q_2 are obtained from the moment equation and the compatibility equation.

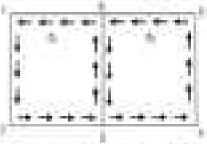


Fig. 6.29 Constant shear flows to be added to the total shear flow.

Moment Equation

The in-plane moment produced by V_1 about any axis must be equal to the in-plane moment about the same axis resulting from the shear flows.
Taking the moment about stringer 1, we have

$$V_1 \cdot 0 = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2 + q'_{23} \cdot 40 \cdot 40 + q'_{34} \cdot 40 \cdot 40$$
$$+ q'_{45} \cdot 40 \cdot 80 - q'_{56} \cdot 40 \cdot 40 \qquad (6.33)$$

Where,

$$\bar{A}_1 = \bar{A}_2 = 40 \cdot 40 = 1600\, cm^2$$

Substituting the numerical values of given by Eqs. (6.31) and (6.32) into Eq. (6.33), we obtain

$$q_1 + q_2 = -80\ N/cm$$

Compatibility Equation

The compatibility condition requires that the twist angle of cell 1 must be equal to that of cell 2. Using Eq. (4.67), we have

$$\frac{1}{2\bar{A}_1 G}\left[\frac{40 q_{61}}{t_1} + \frac{40 q_{12}}{t} + \frac{40 q_{23}}{t_1} - \frac{40 q_{36}}{t_1}\right] \qquad (6.35)$$
$$= \frac{1}{2\bar{A}_2 G}\left[\frac{40 q_{34}}{t_1} + \frac{40 q_{45}}{t} + \frac{40 q_{56}}{t_1} - \frac{40 q_{63}}{t_1}\right]$$

where $\bar{A}_1$ and $\bar{A}_2$ are the areas enclosed by the center lines of the thin walls in Cells 1 and 2, respectively, and

$$q_{[illegible]} = q_1$$

$$q_{[illegible]} = q'_{[illegible]} + q_1 = q_1 - 20$$

$$q_{[illegible]} = q'_{[illegible]} + q_1 = q_1$$

$$q_{[illegible]} = q'_{[illegible]} + q_1 + q_2 = -40 + q_1 + q_2$$

$$q_{[illegible]} = q'_{[illegible]} + q_2 = q_2$$

$$q_{[illegible]} = q'_{[illegible]} + q_2 = -80 + q_2$$

$$q_{[illegible]} = q'_{[illegible]} + q_2 = q_2$$

Equation (6.35) is simplified to

$$10q_1 - 7q_2 = -60$$

Solving Eqs. (6.34) and (6.36), we obtain

$$q_1 = -36.47 \text{ N/cm}, \qquad q_2 = -43.53 \text{ N/cm}$$

Shear Center

To find the shear center, we assume that the applied force passes through the shear center as shown in Figure 6.20.

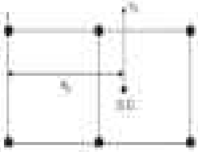


Fig. 6.20 Applied force passing through a shear center

The resultant torque of the shear flow and the torque produced by V_y must be equal. Taking the moment about stringer 1, we have

$$V_y e_x = \sum_{sheet} M_1$$

where ΣM_1 is the same as the right-hand side of (6.33). Explicitly, the equation above is given by

$$V_y e_x = 3200(q_1 + q_2) + 256{,}000$$

By the definition of shear center, we require that

$$\theta_1 = 0 = q_{10} + q_{12} + q_{23} - q_{03}$$

$$\theta_2 = 0 = \frac{q_{20}}{t_2} + \frac{q_{21}}{t_1} + \frac{q_{23}}{t_2} + \frac{q_{03}}{t_2}$$

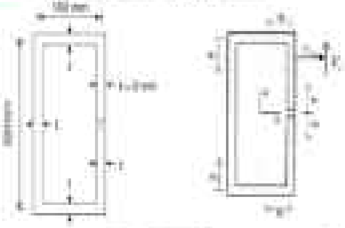
These three equations are sufficient to solve for q_1, q_2, and the shear center location e_1. The solutions are

$$q_1 = -4.44 \text{ N/cm}$$

$$q_2 = 2.22 \text{ N/cm}$$

$$e_1 = 51.85 \text{ cm}$$

Example Find the shear center e_1 for the section:



Solution:

(1) Since the cross-section is symmetric with respect to x axis, shear center is located on the x axis. So it is only necessary to determine the x position of the shear center.

Assume that the transverse shear force V_y and $V_x = 0$ acts through the shear center at the distance e_x to the left/right of the web. The shear flow can be obtained by

$$q_s = -\frac{V_y Q}{I_x} \qquad (5.13)$$

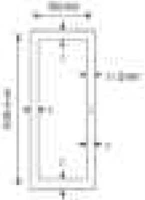
where $Q = \int y\, dA = A'\bar{y}'$ is the first moment of area A' and $\bar{y}'$ is the vertical distance from the centroid of A' to the x axis.

$$I_x = \frac{1}{12}\left[(0.1+0.002)(0.2+0.002)^3 - (0.1-0.002)(0.2-0.002)^3\right]$$
$$= 6.6676 \times 10^{-6}\ m^4$$

On s_1: 0 – 0.1m

$$Q = \int_{A'} y\,dA = \bar{y}'A' = (s_1 t)\left(\frac{s_1}{2}\right) = 0.001 s_1^2$$

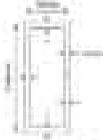
$$q_1 = -\frac{V_y Q}{I_z} = -\frac{V_y \times 0.001 s_1^2}{6.6676\times10^{-6}} = -150 V_y s_1^2$$



On s_2: 0 – 0.2m

$$Q = 0.001(0.1)^2 + 0.002 s_2 (0.1) = 10^{-5} + 2\times10^{-4} s_2$$

$$q_2 = -\frac{V_y Q}{I_z} = -\frac{V_y \times (10^{-5} + 2\times10^{-4} s_2)}{6.6676\times10^{-6}} = -1.5V_y - 30V_y s_2$$



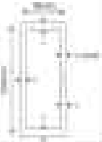
On s_3 : 0 – 0.1m

$$Q = 0.0001(1)^2 + 0.0002(0.1)^2 + 0.001s_3(0.1-0.5s_3)$$

$$= 3\times10^{-7} + 2\times10^{-4}(s_3 - 5s_3^{\,2})$$

$$q_3 = -\frac{V_z Q}{I_y} = -\frac{V_z\times[3\times10^{-7}+2\times10^{-4}(s_3-5s_3^{\,2})]}{6.6676\times10^{-8}}$$

$$= -4.5V_z - 3000V_z s_3 + 15000V_z s_3^{\,2}$$



On s_4 : 0 – 0.1m

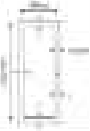
$$Q = \iint_A z\,dA = A_4\bar{z}_4 = (s_4 t)(\frac{s_4}{2}) = -0.001s_4^{\,2}$$

$$q_4 = -\frac{V_z Q}{I_y} = \frac{V_z\times0.001s_4^{\,2}}{6.6676\times10^{-8}} = 15000V_z s_4^{\,2}$$

On s_1 : 0 – 0.1m

$$Q = -0.001(0.1)^2 - 0.002 s_1 (0.1) = -10^{-5} - 2\times10^{-4} s_1$$

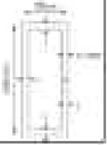
$$q_s = -\frac{V_y Q}{I_z} = \frac{V_y \times (10^{-5} + 2\times10^{-4} s_1)}{6.6676\times10^{-6}} = 1.5V_y + 30V_y s_1$$



On s_2 : 0 – 0.1m

$$Q = -0.001(0.1)^2 - 0.002(0.1)^2 - 0.002 s_2 (0.1 - 0.5 s_2)$$
$$= -3\times10^{-5} - 2\times10^{-4}(s_2 - 5 s_2^2)$$

$$q_s = -\frac{V_y Q}{I_z} = \frac{V_y \times [3\times10^{-5} + 2\times10^{-4}(s_2 - 5 s_2^2)]}{6.6676\times10^{-6}}$$
$$= 4.5V_y + 30V_y s_2 - 150V_y s_2^2$$



Shear center

The moment produced by V_y and the shear flow about the right-top corner must be equal. And we can see that only q_{2s}, q_{3s}, q_{4s} produce the moment with respect to the right-top corner.

i. moment produced by V_y

$M_{0} = V_y x_{sc}$ (counterclockwise)

ii. moment produced by q_{2s}

$$M_{q2} = \int_0^{0.1}(q_{2s})(0.1)ds_2 = \int_0^{0.1}(-4.5V_y - 30V_y s_2 + 150V_y s_2^2)(0.1)ds_2$$

$$= (-0.45V_y s_2 - 1.5V_y s_2^2 + 5V_y s_2^3)|_0^{0.1} = -0.055V_y$$

(counterclockwise)

iii. moment produced by q_{3s}

$$M_{q3} = \int_0^{0.2}(q_{3s})(0.2)ds_3 = \int_0^{0.2}(1.5V_y + 10V_y s_3)(0.2)ds_3$$

$$= (0.3V_y s_3 + 1V_y s_3^2)|_0^{0.2} = 0.1V_y$$

(clockwise)

iv. moment produced by q_{4s}

$$M_{q4} = \int_0^{0.1}(q_{4s})(0.1)ds_4 = \int_0^{0.1}(4.5V_y + 30V_y s_4 - 150V_y s_4^2)(0.1)ds_4$$

$$= (0.45V_y s_4 + 1.5V_y s_4^2 - 5V_y s_4^3)|_0^{0.1} = 0.055V_y$$

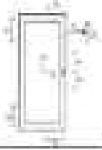
(clockwise)

So we can determine the shear center by $M_P = \sum M_i$

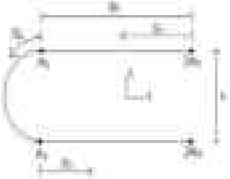
$\Rightarrow V_y x_{sc} = (-0.025V_y)(-0.06V_y) - 0.035V_y$

$\Rightarrow x_{sc} = -0.17m$

The negative sign indicates the opposite side as we assumed.



Example Find the shear center y_s for the sections:



Open thin-walled section

- Since this cross-section is symmetric with respect to y axis, shear center is located in the y axis. So it is only necessary to determine the y position of the shear center.
- Assume that the transverse shear force V_y (and $V_z = 0$) acts through the shear center and produces no twisting in the cross-section. Also assume that the thin sheets are ineffective in bending. Let the left-bottom stringer be the center of origin. We can now determine the centroid of this four-stringer section.

$$\bar{z} = \frac{\sum A_i z_i}{\sum A_i} = \frac{2(2A_1)(20)}{2(A_1)+2(2A_1)} = 1.3333$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{(2A_1)(10)}{2(A_1)+2(2A_1)} = 0.16$$

Now set the y and z axis to match the centroid.

The moment of inertia then can be obtained

$$I_z = \sum A_i y_i^2 = 2(2A_1)(0.5)^2 = 1.5A_1 l^2$$

$$I_y = \sum A_i z_i^2 = 2(2A_1)(0.6667l)^2 + 2(A_1)(1.3333l)^2 = 5.3333A_1 l^2$$

i. For q_1

$Q_1 =$ [illegible]

$q_1 = \frac{V Q_1}{I_x} =$ [illegible]

ii. For q_2

$Q_2 =$ [illegible]

$q_2 = \frac{V Q_2}{I_x} =$ [illegible]

iii. For q_3

$Q_3 =$ [illegible]

$q_3 = \frac{V Q_3}{I_x} =$ [illegible]

The shear flow of the [illegible] section is

$q = \frac{VQ}{I}$

(d) Shear centre

The moment produced by V and the shear flow about the [illegible] must be equal:

i. moment produced by V

$M_o =$ [illegible] (counterclockwise)

ii. moment produced by q_1

$M_o =$ [illegible]

(counterclockwise)

iii. moment produced by q_2

$M_o =$ [illegible]

(counterclockwise)

iv) moment produced by q_4

$$M_{q4} = q_4 (2h)(0.5h) = (-\frac{0.6667 P_y}{h})(2h)(0.5h) = -0.6667 P_y h$$

(counterclockwise)

So we can determine the shear centre by $M_1 = \sum M_q$

$\Rightarrow P_y x_s = -0.6667 P_y h - 0.7854 P_y$ [illegible] $- 0.6667 P_y h = -2.1188 P_y h$

$\Rightarrow x_s = -2.1188h$

The negative sign indicates the opposite side as we assumed.

Example

Find the flexural shear flow in the section of Fig 5.32 for $P_y = 5000$ N.

Open thin-walled section

Solution:

(a) Assume that the transverse shear force acts through the shear center and produces no torsion of the cross-section. One assumes that the thin sheets are ineffective in bending. Let the left-bottom stringer be the origin of a coordinate system with respect to which we now determine the centroid of this four-stringer section. We obtain

$$y_c = \frac{\sum A_i y_i}{\sum A_i} = \frac{[illegible]}{[illegible]} = [illegible]$$

$$z_c = \frac{\sum A_i z_i}{\sum A_i} = \frac{[illegible]}{[illegible]} = [illegible]$$

Now we set up the (y, z) coordinate system with the origin located at the centroid. The moments of inertia with respect to y-axis and z-axis are obtained as

$$I_y = \sum A_i z_i^2 = [illegible]$$

$$I_z = \sum A_i y_i^2 = [illegible]$$

(b) The shear flow calculation

The shear flow formula for symmetric and open sections

$$q = -\frac{V_z Q_y}{I_y} \qquad [illegible]$$

(1) On s_1

$$Q_1 = (2A_f)(0.5h) = A_f h$$

$$q_1 = -\frac{V_y Q_1}{I_x} = -\frac{5000 \times A_f h}{1.5 A_f h^2} = -\frac{3333.33}{h}$$

(2) On s_2

$$Q_2 = (3A_f)(0.5h) = 1.5 A_f h$$

$$q_2 = -\frac{V_y Q_2}{I_x} = -\frac{5000 \times 1.5 A_f h}{1.5 A_f h^2} = -\frac{5000}{h}$$

(3) On s_3

$$Q_3 = (3A_f)(0.5h) - (A_f)(0.5h) = 1A_f h$$

$$q_3 = -\frac{V_y Q_3}{I_x} = -\frac{5000 \times A_f h}{1.5 A_f h^2} = -\frac{3333.33}{h}$$

The negative sign indicates that the actual shear flow direction is opposite to the assumed direction of the contour.

Find the shear flow for the three-stringer section shown in Figure 4.34 for S_y = 5000 N and S_x = 0. Given shear modulus G = 27 GPa, find the twist angle per unit length. Also determine the shear centre. Is the shear flow statically determinate?

Example

Fig. 4.34 Single-cell closed section.

Solution:

(a) Assume that the thin sheets are ineffective in bending. Let stringer 2 be the reference point for the location of the centroid of the three-stringer section. We have the horizontal and vertical distances of the centroid from stringer 2 as:

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{[illegible]}{3(A)} = [illegible]$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{[illegible] + [illegible]}{3(A)} = [illegible]$$

Now we set up the y and z axes with the origin at the centroid. The moments of inertia are:

$$I_y = \sum_i A_i z_i^2 = 2(10\times10^{-4})(0.1)^2 = 2\times10^{-5}\ m^4$$

$$I_z = \sum_i A_i y_i^2 = 2(10\times10^{-4})(0.2667)^2 + (10\times10^{-4})(0.4-0.2667)^2$$

$$= 4.2667\times10^{-5}\ m^4$$

(b) Shear flows

Since this cross-section is symmetric with respect to y axis, the shear center is located on the y axis. Hence only the z position of the shear center needs to be found. We first make a fictitious cut between stringers 1 and 3 and consider the shear flow as the superposition of two shear flow systems as shown in the figure below.

(1) First, calculate the shear flows by assuming a cut in the wall between stringers 1 and 2. Then $q'_{12}=0$, and the shear in the cut section is calculated by using

$$q_s = -\frac{V_y Q_x}{I_x} \qquad (5.5.6)$$

where $Q_x = \sum_{i=1}^{n} A_i y_i$ is the first moment of stringer area A_i and y_i is the vertical distance from the centroid of A_i to the x-axis. We obtain

$$q'_{23} = -\frac{V_y Q_x}{I_x} = -\frac{5000(10\times10^{-4}\times0.1)}{2\times10^{-5}} = -25000\ N/m$$

$$q'_{34} = -\frac{V_y Q_x}{I_x} = -\frac{5000(10\times10^{-4}\times0.1 - 10\times10^{-4}\times0.1)}{2\times10^{-5}} = 0$$

(2) Adding the shear flow q_0 from the second part, we form the total shear flow as

$q_{23} = q'_{23}+q_0 = -25000+q_0$

$q_{34} = q'_{34}+q_0 = q_0$

$q_{41} = q'_{41}+q_0 = q_0$

The resulting moment of the total shear flow must be equal to the moment produced by V_y. Taking moment about stringer 1, we have:

$V_y \times 0 = 2A_{23}q_{23} + 2A_{34}q_{34}$

$= 2\frac{\pi}{4}(0.1^2)(-25000+q_0) + 2\frac{(0.2)(0.2)}{2}q_0$

$= -392.6 + 0.1914 q_0 = 0$

$\Rightarrow q_0 = 11810\ N/m$

Note: $\bar{A}_I$ is the area enclosed by the curved sheet and the line connecting stringers 1 and 2. $\bar{A}_{II}$ is the area enclosed by the lines connecting stringers 1, 2 and 3. Also note that the shear flow passes stringer 1 and, thus, does not produce any moment.

The final shear flows are

$$q_{12} = -25000 + q_0 = -20897 N/m$$

$$q_{23} = q_0 = 4103 N/m$$

$$q_{31} = q_0 = 4103 N/m$$

(ii) Twist angle per unit length

The equation of the twist angle per unit length is

$$\theta = \frac{1}{2GA}\oint \frac{q}{t} ds$$

We can obtain the twist angle per unit length as

$$\theta = \frac{1}{2GA}\oint \frac{q}{t} ds = \frac{-20897 \times \frac{0.2\pi}{2} + 2 \times 4103 \times (0.001)}{2 \times 27 \times 10^9 \times (\frac{\pi}{8} \times 0.2^2 + 0.2 \times 0.4 / 2)(0.001)} = \frac{31.34}{3.17 \times 10^6}$$

$$= 9.87 \times 10^{-6} rad/m = 5.7 \times 10^{-4} deg/m$$

(4) Shear centre

To determine the horizontal location of the shear centre, we assume that the shear force V_y acts through the shear centre which is assumed to be located at a horizontal distance x to the right of stringer 1. We can write the moment equation as

[illegible]

We can solve q_1 in terms of x

$\Rightarrow q_1 = 20123.58x - [illegible]$

So that the shear flows within each sheet are

$q_{12} = q'_{12} + q_1 = -25000 + q_1 = 20123.58x - [illegible]$

$q_{23} = q'_{23} + q_1 = 20123.58x - 1100.00$

$q_{31} = q'_{31} + q_1 = 20123.58x - [illegible]$

Since the shear force V_y passes through the shear centre, the twist angle is equal to zero, i.e. $\theta = \frac{1}{2GA}\oint \frac{q}{t} ds = 0$

$\Rightarrow \frac{1}{2GA}\left([illegible] \right) = 0$

$$\frac{1}{2A}\left[\ldots\right]=0$$

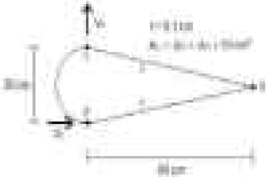
$\Rightarrow$ [illegible]

$\Rightarrow x=-[illegible]$

The negative sign means that the shear centre is to the left of stringer 1.

Example

[illegible]



$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = [illegible]$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = [illegible]$$

The moments of inertia are obtained as:

$$I_x = \sum A_i y_i^2 = [illegible]$$

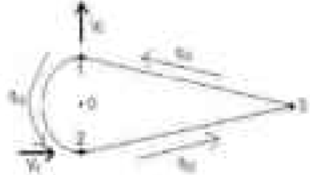
$$I_y = \sum A_i x_i^2 = [illegible]$$

$$= [illegible]$$

(b) Shear flows

We can solve the problem by considering the two applied forces separately and then superposed the two solutions. Alternatively, we can calculate the shear flows by taking advantage of the fact that they are statically determinate shear flows.

Denote the shear flows by q_{12}, q_{23}, q_{31} respectively, in the sheet between stringers 1 and 2, stringers 2 and 3, and stringers 3 and 1, as shown in the figure below.



Using the condition that the resultants of the shear flows must be equal to the applied shear forces, we have:

$$\sum F_x = V_x \Rightarrow q_{23}(0.8) - q_{31}(0.8) = 10000$$

$$\sum F_y = V_y \Rightarrow q_{23}(0.1) + q_{31}(0.1) - q_{12}(0.2) = 5000$$

$\sum M_o =$ moment about z-axis passing through o

$$\Rightarrow q_{12}\left(\frac{\pi (0.2)^2}{4}\right) + (q_{23} + q_{31})(0.8)(0.1) = 5000 \times 0 + 10000 \times 0.1$$

Solving the equations (5.6.1) to (5.6.3), we obtain:

$q_{12} = -15673\ N/m$

$q_{23} = 15577\ N/m$

$q_{31} = 3077\ N/m$

(c) Twist angle per unit length

The equation for twist angle per unit length is

$$\theta' = \frac{1}{2GA}\oint \frac{q}{t}\,ds$$

= [illegible]

(d) Shear center

[illegible]

Example: Find the shear flow q in the four-stringer section (Fig. 5.34) subjected to $V_z = 10000$ N, assume that the thin skin is ineffective in bending.

$t = 0.2$ cm
$A_1 = A_4 = 15\ cm^2$
$A_2 = A_3 = 10\ cm^2$

Figure 5.34 Unsymmetrical open section

Solution:

(a) The centroid of this four-stringer section is located at

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{2(10)(80)}{2(15)+2(10)} = 32cm$$

$$\bar{z} = \frac{\sum A_i z_i}{\sum A_i} = \frac{(15)(40)+(10)(20)}{2(15)+2(10)} = 16cm$$

from stringer 2.

Now set up the (y, z) coordinate system with the origin at the centroid. The moments of inertia are obtained as

$$I_y = \sum A_i z_i^2 = [illegible]$$

$$= [illegible]\ cm^4$$

$$I_z = \sum A_i y_i^2 = [illegible]$$

$$= [illegible]\ cm^4$$

$$I_{yz} = \sum A_i y_i z_i = [illegible]$$

$$= [illegible] = -[illegible]\ cm^4$$

(b) Shear flows

Assume the shear force acts through the shear centre and no torsion is produced. The shear flow in the unsymmetrical thin-walled section is calculated using

$$q_s = -(k_y V_y - k_{yz} V_z) Q_z - (k_z V_z - k_{yz} V_y) Q_y$$

$$\text{where } k_y = \frac{I_y}{I_y I_z - I_{yz}^2},\ k_z = \frac{I_z}{I_y I_z - I_{yz}^2},\ k_{yz} = \frac{I_{yz}}{I_y I_z - I_{yz}^2}$$

$$\text{and } Q_z = \iint_{A_s} y\, dA,\ Q_y = \iint_{A_s} z\, dA$$

In the problem, we have

$$k_y = \frac{I_y}{I_y I_z - I_{yz}^2} = \frac{15200}{(15200)(76000) - (-9600)^2} = 1.430\times10^{-5}$$

$$k_z = \frac{I_z}{I_y I_z - I_{yz}^2} = \frac{76000}{(15200)(76000) - (-9600)^2} = 7.149\times10^{-5}$$

$$k_{yz} = \frac{I_{yz}}{I_y I_z - I_{yz}^2} = \frac{-9600}{(15200)(76000) - (-9600)^2} = -9.031\times10^{-6}$$

Using $V_z = -5000$, $V_y = 0$, we have from (2.7.1)

$$q_s = -(k_y V_z - k_{yz} V_y)Q_y - (k_z V_y - k_{yz} V_z)Q_z$$
$$= -6.849\times10^{-2} Q_y - 38.7149\times10^{-3} Q_z$$

The shear flow contour direction is indicated in the figure below.

(1) For shear flow q_1

$Q_x = (15)(-32) = -480cm^3$

$Q_y = (15)(24) = 360cm^3$

From equation (5.7.2) we have

$q_1 = -4.4643\times10^{-2}(-480) - 35.7143\times10^{-2}(360)$
$= -107.14 \ N/cm$

(Negative sign indicates the opposite direction)

(2) For shear flow q_2

$Q_x = 2(15)(-32) = -960cm^3$

$Q_y = (15)(24-16) = 120cm^3$

From equation (5.7.2) we have

$q_2 = -4.4643\times10^{-2}(-960) - 35.7143\times10^{-2}(120) = 0$

(7) For shear flow q_1:

$Q_x = 2(13)(-12) + (0)(0)(40) = -480 cm^3$

$Q_y = (2.5)(24-16) + (0)(0-10) = -40 cm^3$

From equation (5.7.2) we have

$q_1 = -4.06(5)\times10^{-3}(-480) - (3.32)(5)\times10^{-3}(-40)$
$= 13.91 \ N/cm$

Example Find the shear center (x_s, y_s) for the open section in Fig. 5.14.

Figure 5.14 Unsymmetrical open section

Solution:

(a) Assume that the thin sheets are ineffective in bending. The centroid of the [illegible] section is located at

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{[illegible]}{2(175)+2(100)} = 32mm = 0.32m$$

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{[illegible]}{2(175)+2(100)} = [illegible]$$

[illegible] 2

Now set up the x_1, y_1 coordinate system with the origin at the centroid as shown in Fig. 7.18.

The moments of inertia are

$$I_x = \sum A_i y_i^2 = [illegible]$$

$$= 1.52 \times 10^{-4} m^4$$

$$I_y = \sum A_i x_i^2 = [illegible]$$

$$= [illegible]$$

$$I_{xy} = \sum A_i x_i y_i = [illegible]$$

$$= [illegible]$$

(b) Shear Flow:

Assume the shear force acts through the shear center and no torsion is produced. The shear flow in an unsymmetrical thin-walled section is calculated using the equation

$$q_s = -(k_x S_x - k_{xy} S_y) Q_y - (k_y S_y - k_{xy} S_x) Q_x$$

where $k_x = \frac{I_x}{I_x I_y - I_{xy}^2}$; $k_y = \frac{I_y}{I_x I_y - I_{xy}^2}$; $k_{xy} = \frac{I_{xy}}{I_x I_y - I_{xy}^2}$

and $Q_y = \int_0^s x t\, ds$, $Q_x = \int_0^s y t\, ds$

In this problem, we have:

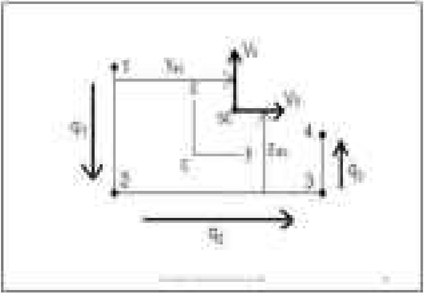
$$k_x = \frac{I_x}{I_x I_y - I_{xy}^2} = \frac{[illegible]}{[illegible]} \times 10^{[illegible]} = [illegible]$$

$$k_y = \frac{I_y}{I_x I_y - I_{xy}^2} = \frac{[illegible]}{[illegible]} \times 10^{[illegible]} = [illegible]$$

$$k_{xy} = \frac{I_{xy}}{I_x I_y - I_{xy}^2} = \frac{[illegible]}{[illegible]} \times 10^{[illegible]} = [illegible]$$

Thus equation (7.9) becomes:

$$q_s = -(k_x S_x - k_{xy} S_y) Q_y - (k_y S_y - k_{xy} S_x) Q_x$$
$$= -([illegible] S_x + [illegible] S_y) \times 10^{[illegible]} Q_y - ([illegible] S_y + [illegible] S_x) \times 10^{[illegible]} Q_x \qquad ([illegible])$$



Assume that the shear centre is located at (y_s, z_s) with respect to stringer 2 as shown in the figure above.

(1) For shear flow q_1

$Q_z = (1\times10^{-4})(-0.5) = -4.8\times10^{-5}\ m^3$

$Q_y = (1\times10^{-4})(0.2) = 5.0\times10^{-5}\ m^3$

From equation (5.8.2) we have

$$q_1 = -(1.4017V_y + 0.0028V_z)\times10^{3}Q_z - (7.1429V_z + 0.0028V_y)\times10^{3}Q_y$$
$$= -(1.4017V_y + 0.0028V_z)(-4.8)\times10^{-2} - (7.1429V_z + 0.0028V_y)(5.0)\times10^{-2}$$
$$= 0.1177V_y - 2.1AV_z$$

(2) For shear flow q_2

$\bar{Q}_1 = $ [illegible]

$\bar{Q}_2 = $ [illegible]

From equation (7.6.2) we have:

$q_2 = $ [illegible]

$= $ [illegible]

$= $ [illegible]

(3) For shear flow q_3

$\bar{Q}_1 = $ [illegible]

$\bar{Q}_2 = $ [illegible]

From equation (7.6.2) we have:

$q_3 = $ [illegible]

$= $ [illegible]

$= $ [illegible]

Check balance of forces:

[illegible]

[illegible]

(c) Shear centre [illegible]

(i) To determine [illegible], we consider the case [illegible] and [illegible], and consider moment about stringer [illegible]:

[illegible]

[illegible]

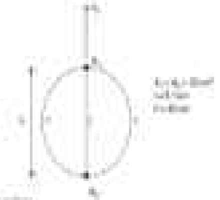
(ii) For [illegible] we consider the case [illegible] and [illegible]. We have:

[illegible]

[illegible]

So the shear centre is located at [illegible] with respect to stringer [illegible]. This says that the shear centre is [illegible] between stringers [illegible] and [illegible].

Example [illegible]

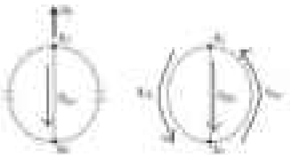


[illegible]

Solution:

[illegible]

(b) Shear flows



Cuts are made on the curved webs as shown in the above figure. Note that the cut section is basically reduced to a single vertical web and thus, the shear flows are simply:

$$q'_{13} = q'_{23} = 0$$

$$q'_{12} = \frac{V_y}{h} = \frac{5000}{0.4} = -12{,}500\ N/m$$

So the shear flows in the original section are

$$q_{12} = q'_{12} - q_2 + q_1 = -12500 - q_2 + q_1$$

$$q_{13} = q_1$$

$$q_{23} = q_2$$

where q_1 and q_2 are the constant shear flows in the left and the right cells, respectively. Note that q_1 and q_2 are both assumed to be positive if counterclockwise.

(1) Moment equation

Take the moment about the centroid of the cross-section, we have

$$T_z = 0 = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2$$

$$\text{where } \bar{A}_1 = \bar{A}_2 = \frac{1}{8}\pi h^2 = 0.0625 m^2$$

$$\Rightarrow q_1 + q_2 = 0$$

(2) Compatibility equation

$$\theta = \frac{1}{2GA}\oint \frac{q}{t}\, ds$$

We have $\theta = \theta_1 = \theta_2$

$$\frac{1}{2G\bar{A}_1 t}[(q_1)\pi\frac{h}{2} + (q_1 - q_2)h] = \frac{1}{2G\bar{A}_2 t}[(q_2 - q_1)h + (q_2)\pi\frac{h}{2}]$$

$$\Rightarrow 0.6283 q_1 - 0.6283 q_2 = -10000 - 0.8 q_1 + 0.8 q_2 \quad \text{[illegible]}$$

$$\Rightarrow q_1 - q_2 = -7800 (N/m) \quad \text{[illegible]}$$

Solving equations (3.18.2) and (3.18.3), we have

$q_1 = -5300 \ (N/m)$

$q_2 = 3500 \ (N/m)$

Then the final shear flows are:

$q_{12} = -12100 - q_1 + q_2 = -3400 \quad N/m$

$q_{23} = q_1 = -5300 \quad N/m$

$q_{34} = q_2 = 3500 \quad N/m$

Negative sign means the actual direction is opposite to the assumed.

And the twist angle is

$$\theta = \theta_1 = \frac{1}{2GA_1}\oint \frac{q}{t}\, ds = \frac{[illegible]}{2.27 \times 10^{?}\,[illegible]}$$

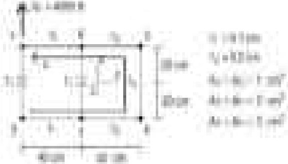
$= 0 \ (rad/m)$

$= 0^{\circ}/m$

As expected, there is no twist angle produced since the vertical load is applied through the shear centre.

Example

[illegible]

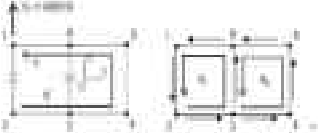


Solution:

(a) [illegible]

[illegible]

(b) Shear flows



The shear flows in the cut section as shown in the left figure above are calculated first. We have

$$q'_{2} = -\frac{V_y Q}{I_x} = -\frac{4000(-20)}{4000} = 20\ N/cm$$

$$q'_{3} = -\frac{4000(1+2)(-20)}{4000} = 60\ N/cm$$

$$q'_{4} = -\frac{4000(1+2+3)(-20)}{4000} = 120\ N/cm$$

$$q'_{5} = -\frac{4000[(1+2+3)(-20)+(3)(20)]}{4000} = 60\ N/cm$$

$$q'_{6} = -\frac{4000[(1+2+3)(-20)+(3+2)(20)]}{4000} = 20\ N/cm$$

Thus, the shear flows are given by

$$q_{12} = q_1$$

$$q_{23} = q'_{23} + q_1 = 20 + q_1$$

$$q_{34} = q_1 - q_L$$

$$q_{45} = q'_{45} + q_1 = 60 + q_1$$

$$q_{56} = q'_{56} + q_1 = 120 + q_1$$

$$q_{67} = q'_{67} + q_1 = 60 + q_1$$

$$q_{78} = q'_{78} + q_1 = 20 + q_1$$

in which q_1 and q_L are the two unknown constant shear flows in the left and right cells of the original closed section.

(2) Moment equation

The moment produced by the shear force [illegible] must be equal to that produced by the shear flows. Taking moment about stringer 1, we have:

[illegible]

[illegible] (3.12.2)

(3) Compatibility equation

[illegible]

We have $\theta = \theta_1 = \theta_2$

[illegible]

[illegible]

[illegible]

Solving equations (3.12.1) and (3.12.3) we have:

[illegible]

[illegible]

Then the shear flows are:

$$q = q_1 = -56.47 \quad N/cm$$

$$q = 20 + q_1 = -36.47 \quad N/cm$$

$$q = q_1 - q_2 = 47.06 \quad N/cm$$

$$q = 60 + q_2 = -43.53 \quad N/cm$$

$$q = 120 + q_2 = 16.47 \quad (N/cm)$$

$$q = 60 + q_2 = -43.53 \quad N/cm$$

$$q = 20 + q_2 = -36.47 \quad N/cm$$

The negative sign means the actual shear flow is opposite to the assumed direction.

Find the shear flow in the two-cell thin-walled section for $V_y = 2000$ N shown in Figure. Also determine the shear center. Assume thin sheets to be ineffective in bending.

Two cell closed section.

Solution:

(a) The centroid of this three-stringer section is

$$y_c = \frac{\sum A_i y_i}{\sum A_i} = \frac{2A(40)}{3A} = 26.67cm$$

$$z_c = \frac{\sum A_i z_i}{\sum A_i} = \frac{A(20)}{3A} = 6.67cm$$

relative to stringer 1. Set up the (y,z) coordinate system with the origin placed at the centroid.

The moments of inertia are:

$$I_y = \sum A_i z_i^2 = 2(10)\left(-\frac{20}{3}\right)^2 + (10)\left(\frac{40}{3}\right)^2 = 2666.67cm^4$$

$$I_z = \sum A_i y_i^2 = 2(10)\left(\frac{40}{3}\right)^2 + (10)\left(-\frac{80}{3}\right)^2 = 10666.67cm^4$$

$$I_{yz} = \sum A_i y_i z_i = 2666.67cm^4$$

(b) Shear flow:

[illegible] (5.15 [illegible])

$$q_s = -(k_x V_x - k_{xy} V_y) Q_y - (k_y V_y - k_{xy} V_x) Q_x$$

where $k_x = \frac{I_x}{I_x I_y - I_{xy}^2}$, $k_y = \frac{I_y}{I_x I_y - I_{xy}^2}$, $k_{xy} = \frac{I_{xy}}{I_x I_y - I_{xy}^2}$

and $Q_x = \iint y\,dA$, $Q_y = \iint x\,dA$

In this problem, we have:

$k_x = \frac{I_x}{I_x I_y - I_{xy}^2} = 1.25\times10^{-4}\ \text{cm}^{-4}$, $k_y = \frac{I_y}{I_x I_y - I_{xy}^2} = 5\times10^{-5}\ \text{cm}^{-4}$

$k_{xy} = \frac{I_{xy}}{I_x I_y - I_{xy}^2} = 1.25\times10^{-5}\ \text{cm}^{-4}$, $V_y = 50000$, $V_x = 0$

Then equation (5.15.1) becomes

$$q_s = -(I_x F_y - I_{xy} F_x) Q_x - (I_y F_x - I_{xy} F_y) Q_y$$

$$= 0.0625 Q_x - 2.5 Q_y$$

(1) For shear flow q_{12}

$$Q_x = (10 \times \frac{80}{3}) = -266.67 cm^3$$

$$Q_y = (10 \times \frac{20}{3}) = -66.67 cm^3$$

From equation (5.15.4) we have

$$q_{12} = 0.0625(-266.67) - 2.5(-66.67) = 0$$

(2) For shear flow q'_{23} (in the curved sheet)

$$Q_x = -133.33 cm^3$$

$$Q_y = -133.33 cm^3$$

From equation (5.15.4) we have

$$q'_{23} = 0.0625(-133.33) - 2.5(-133.33) = 2500 (N/cm)$$

The total shear flow is obtained by adding the constant counterclockwise shear flows q_1 and q_2 in the left and right cells, respectively. Thus,

$$q_{12} = q'_{12} + q_1 = q_1$$

$$q_{14} = q_1 - q_2 \quad \text{(vertical sheet)}$$

$$q_{34} = q'_{34} + q_2 = 250 + q_2$$

$$q_{41} = q_1$$

(1) Moment equation

Take moment about stringer 1. We have

$$T_x = 0 = 2q_{34}\left[\left(\frac{\pi}{8}20^2\right) + \frac{1}{2}\times 20\times 40\right] + 2q_{12}\times\frac{1}{2}\times 20\times 40$$

$$\Rightarrow 800q_1 + 1114.16q_2 = -278540$$

(2) Compatibility equation

$$\theta = \frac{1}{2A_i G t} \oint q \, ds$$

We have $\theta = \theta_1 = \theta_2$

$$\frac{1}{2G A_1}[\text{[illegible]}] = \frac{1}{2G A_2}[\text{[illegible]}]$$

$\Rightarrow 135.69q_1 - 153.59q_2 = 20000$ [illegible]

Solve equations (5.15.5) and (5.15.6), we have

$q_1 = -210.78 (N/cm)$

$q_2 = -349.86 (N/cm)$

Then the shear flows in each sheet are

$q_{12} = -210.78 (N/cm)$

$q_{2m} = 139.08 (N/cm)$

$q_{23} = -89.08 (N/cm)$

$q_{31} = -210.78 (N/cm)$

Negative sign means the actual direction of the shear flow is opposite to the assumed.

Check:

$\sum F_x = $ [illegible] $= 0$

$\sum F_y = $ [illegible] $= 5000 \; N$

(c) Shear center

Assume the shear force is acting through the shear center which is assumed to be at a distance x_s to the right of stringer 3.

(1) Moment equation

Take moment about stringer 3. We have

$$5000(x_s) = 2q_{12}\frac{1}{2}(40)(20) + 2q_{34}\left(\frac{\pi}{8}20^2\right)$$

$$\Rightarrow 800q_1 + 314.16q_2 = 5000x_s - 25000\pi$$

(2) Compatibility equation

$$\theta = \frac{1}{2GA}\oint \frac{q}{t}ds$$

We have $\theta = \theta_1 = \theta_2 = 0$ when the force acts through the shear center, then

$$\frac{1}{2GA_1}([illegible]) = 0$$

$\Rightarrow$ [illegible] (5.15.8)

and $\frac{1}{2GA_2}([illegible]) = 0$

$\Rightarrow$ [illegible] (5.15.9)

Solving equations (5.15.8) and (5.15.9) we have:

Solving equations (5.15.8) and (5.15.9), we have

$q_1 = -31.32 \ (N/cm)$

$q_2 = -165.01 \ (N/cm)$

Substituting back to equation (5.15.7), we obtain

$e_s = 0.3cm$ (to the right of stringers 2 and 3)

Elastic Buckling

Failure in a structure can be classified in two general categories, **material failure** and **structural failure**.

The former includes **plastic yielding, rupture, fatigue and unstable crack growth** (fracture).

In the latter category, examples include **flutter** (excessive dynamic deflection of a structure in airflow) and **buckling**.

Structural failure results in the loss of the designed structural functions and may lead to eventual material failure.

ELASTIC BUCKLING OF STRAIGHT BARS

Consider a centrically ($e = 0$) compressed straight bar. The boundary conditions at the two ends are arbitrary. We want to examine for a given P whether it is possible to maintain a transverse deflection $w(x)$.

If such deflection is possible, then it must satisfy the equilibrium equation and the specified boundary conditions.

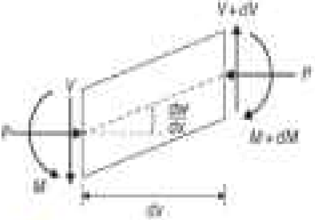


Fig. 8.1 Compressed bar in the buckled position.

Take a differential element from the compressed bar in the (assumed) buckled position as shown in Figure 8.2.

The equilibrium equations for this free body are:

$$\sum F_y = 0: \quad (V + dV) - V = 0$$

or

$$\frac{dV}{dx} = 0 \qquad (8.11)$$

$$\sum M = 0: \quad (V + dV)\,dx + P\frac{dv}{dx}\,dx = dM$$

or

$$V = \frac{dM}{dx} - P\frac{dv}{dx} \qquad (8.12)$$

Substituting Eq. (8.12) into Eq. (8.11), we obtain the equilibrium equation for the assumed deflection:

$$\frac{d^2M}{dx^2} - P\frac{d^2v}{dx^2} = 0 \qquad (8.13)$$

Substituting the relation

$$M = -EI\frac{d^2v}{dx^2} \qquad (4.16)$$

into Eq. (8.13), we obtain the equilibrium equation in terms of deflection as

$$\frac{d^4 w}{dx^4} + k^2 \frac{d^2 w}{dx^2} = 0 \qquad (8.15)$$

$$k = \sqrt{\frac{P}{EI}}$$

The general solution for Eq. (8.15) is readily obtained as

$$w = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4 \qquad (8.16)$$

The four arbitrary constants C_1–C_4 are to be determined by using the boundary conditions.

Pinned–Pinned Bar

Consider a straight bar with pinned ends as shown in Figure 8.2. The boundary conditions are

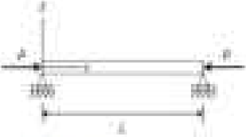


Fig. 8.2 Straight bar with pinned ends.

at $x = 0$: $\quad w = 0,$

$$M = -EI\frac{d^2 w}{dx^2} = 0$$

and

at $x = L$: $\quad w = 0,$

$$M = -EI\frac{d^2 w}{dx^2} = 0$$

Substitution of Eq. (8.16) into Eq. (8.17) yields:

$$C_2 + C_4 = 0 \qquad (8.18a)$$

$$C_2 = 0 \qquad (8.18b)$$

$$C_1 \sin\lambda L + C_2 \cos\lambda L + C_3 L + C_4 = 0 \qquad (8.18c)$$

$$C_1 \lambda^2 \sin\lambda L + C_2 \lambda^2 \cos\lambda L = 0 \qquad (8.18d)$$

From Eqs. (8.18a) and (8.18b), we have $C_2 = C_4 = 0$. Thus, the four equations reduce to:

$$C_1 \sin\lambda L + C_3 L = 0 \qquad (8.19a)$$

$$C_1 \sin\lambda L = 0 \qquad (8.19b)$$

from which we obtain $C_3 = 0$ and

$$C_1 \sin\lambda L = 0 \qquad (8.20)$$

Since $C_1 \neq 0$ (otherwise, we have a trivial solution, i.e. $w = 0$ everywhere), we must require that:

$$\sin\lambda L = 0 \qquad (8.21)$$

Equation (8.21) is satisfied if

$$kL = n\pi, \qquad n = 1, 2, 3, \ldots \qquad (8.22)$$

The corresponding P's that satisfy (8.22) are

$$P_{cr}^{(n)} = \frac{n^2 \pi^2 EI}{L^2}, \qquad n = 1, 2, 3, \ldots \qquad (8.23)$$

The deflection (**buckling mode shape**) for each **critical load** $P_{cr}^{(n)}$ (also called **buckling load**) is

$$v^{(n)}(x) = C_1 \sin k^{(n)} x \qquad (8.24)$$

where

$$k^{(n)} = \sqrt{\frac{P_{cr}^{(n)}}{EI}} \qquad (8.25)$$

Hence, there are infinitely many possible deformed configurations given by Eq. (8.24) that are associated with the axial loads given by Eq. (8.23).

In other words, at the compressive load $P_{cr}^{(n)}$, besides the straight position, the bar can also assume a deformed position given by Eq. (8.24) for any value of C_1.

These values of P, the critical loads, are also called **bifurcation points**.

Among all the critical loads, the lowest one with $n = 1$ is of particular importance because as compression is applied to the bar the lowest buckling load is reached first.

The lowest buckling load (for the first buckling mode with $n = 1$) is

$$P_{cr} = P_{cr}^{(1)} = \frac{\pi^2 EI}{L^2} \qquad (8.26)$$

which is known as **Euler's formula** for column buckling. The corresponding mode shape is

$$w(x) = C_1 \sin \frac{\pi x}{L} \qquad (8.27)$$

To produce the second buckling mode, the first mode must be suppressed. This can be realised by adding a support at the midspan of the bar as shown in Figure 8.4.
In view of the buckling mode shape [Eq. (8.27)], we must set $C_1 = 0$ for the first mode to satisfy the condition $w = 0$ at $x = L/2$. Thus, the first buckling mode is suppressed.

Consider the second buckling mode ($n = 2$), for which the buckling load is

$$P_{cr}^{(2)} = \frac{4\pi^2 EI}{L^2} \qquad (8.28)$$

and the corresponding buckling mode shape is given by:

$$w(x) = C_1 \sin\frac{2\pi x}{L} \qquad (8.21)$$

Thus, the lowest compressive load that can cause buckling of the bar of Figure 8.4 is that given by Eq. (8.20). This load is four times the buckling load for the bar without support at the midspan.

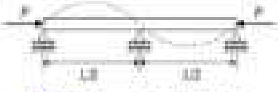


Fig. 8.4 Added support at the midpoint of a bar

Example

An AISI 4340 steel box beam has a length of 3 m and is pinned at both ends. If the cross-sectional area has the dimensions shown, determine the critical compressive load (Figure 8.5).

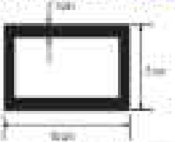


Fig. 8.5 The cross-section of an AISI 4340 steel box beam subjected to compressive load

This is a pinned–pinned bar. As such, the critical buckling load will be given by Eq. (9.23).

$$I_{horizontal} = \frac{1}{12}(0.1)(0.03^3) - \frac{1}{12}(0.08)(0.03^3) = 0.86167 \times 10^{-8}\ m^4$$

$$I_{vertical} = \frac{1}{12}(0.03)(0.1^3) - \frac{1}{12}(0.03)(0.08^3) = 2.00667 \times 10^{-6}\ m^4$$

It is obvious that the critical compressive load will be minimum when the I is minimum.

As such,

$$P_{cr} = P_{cr}^{1} = \frac{\pi^2(200)(10^9)(0.86167)(10^{-8})}{5^2} = 68\ kN$$

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{68000}{(0.1)(0.025) - (0.08)(0.015)} = 26.23\ MPa$$

The compressive strength of AISI 4140 steel is 1,480 MPa. As such, the structure will buckle when the compressive load exceeds 68 kN.

The truss structure consists of two bars connected by a pin joint (which allows free rotation of the bars). The other ends of the bars are hinged as shown in Fig. 7.26. A weight W is hung at the joint. Find the maximum weight the truss can sustain before buckling occurs.

Figure 7.26 [illegible]

Solution:

From equilibrium, axial forces of the bars can be easily determined as:

$$N_1 = \frac{W}{\sin\theta} \quad \text{(Tension)}$$

[illegible]

$$N_2 = -N_1 \cos\theta = -\frac{W\cos\theta}{\sin\theta} \quad \text{(Compression)}$$

Only the compressed bar 2 may suffer buckling when the weight W increases. Since bar 2 is connected with pins at both ends, its buckling load is:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

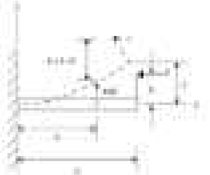
When the axial force N_2 reaches the critical P_{cr} buckling occurs. That is

$$W_{max}\frac{\cos\theta}{\sin\theta} = \frac{\pi^2 EI}{L^2}$$

$$\rightarrow W_{max} = \frac{\pi^2 EI}{L^2}\tan\theta$$

Clamped-Free Bar

The clamped-free bar shown in Figure [illegible] is assumed to be axially compressed ($p > 0$). The boundary conditions are:



$$\text{at } x = 0: \quad w = 0$$

$$\frac{dw}{dx} = 0 \qquad \text{(C27)}$$

$$\text{at } x = L: \quad M = -EI\frac{d^2w}{dx^2} = 0$$

$$V = \frac{dM}{dx} - P\frac{dw}{dx} = 0$$

The last equation can be written as:

$$\frac{d^3w}{dx^3} + k^2\frac{dw}{dx} = 0 \qquad \text{(C28)}$$

Substituting the general solution (8.11) into Eq (8.28), we obtain

$$C_2 + C_4 = 0$$

$$C_1 k + C_3 = 0 \qquad (8.31)$$

$$C_1 k^2 \sin kL + C_2 k^2 \cos kL = 0$$

$$C_3 k^2 = 0$$

From Eqs (8.31b) and (8.31d), we have $C_1 = C_3 = 0$. From Eq (8.31c), we obtain

$$C_2 \cos kL = 0 \qquad (8.32)$$

For a nontrivial solution $C_2 \neq 0$, and we require that

$$\cos kL = 0 \quad \text{or} \quad k = \frac{n\pi}{2L}, \qquad n = 1, 3, 5, \ldots$$

which yields the buckling loads

$$P_{cr}^{(n)} = \frac{n^2 \pi^2 EI}{4L^2}, \qquad n = 1, 3, 5, \ldots$$

The lowest buckling load is

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad (4.28)$$

which is only one-fourth that for the pinned-pinned bar.

The lowest buckling mode shape is

$$v(x) = C_1 \cos kx + C_2 = C_1(\cos kx - 1)$$

which is identical to that of the eccentrically loaded bar (see Figure 4.11) except for the constant amplitude.

Clamped-Pinned Bar

The boundary conditions for the clamped-pinned ends

at $x = 0$: $v = 0$, $\frac{dv}{dx} = 0$

at $x = L$: $v = 0$, $\frac{d^2v}{dx^2} = 0$

Fig. 4.x Bar with clamped-pinned ends.

which yield the following four equations:

$$C_2 + C_4 = 0 \qquad (7.38a)$$

$$C_1 k + C_3 = 0 \qquad (7.38b)$$

$$C_1 \sin kL + C_2 \cos kL + C_3 L + C_4 = 0 \qquad (7.38c)$$

$$C_1 \sin kL + C_2 \cos kL = 0 \qquad (7.38d)$$

Eliminating C_3 and C_4 from (7.38c) and (7.38d) using (7.38a) and (7.38b) we obtain

$$C_1(\sin kL - kL) + C_2(\cos kL - 1) = 0 \qquad (7.39a)$$

$$C_1 \sin kL + C_2 \cos kL = 0 \qquad (7.39b)$$

It is easy to verify that neither $\sin kL = 0$ nor $\cos kL = 0$ can satisfy (7.39) simultaneously.

From (7.39b), we have $\quad C_2 = -\dfrac{\sin kL}{\cos kL} C_1 \qquad (7.40)$

Substituting (7.40) into (7.39a) yields

$$C_1(\tan kL - kL) = 0 \qquad (7.41)$$

For a nontrivial solution, we require

$$\tan kL - kL = 0$$

The solution for kL to the above equation can only be solved numerically. The lowest value that satisfies (7.42) is approximately

$$kL = 4.49 \qquad (7.43)$$

from which the lowest buckling load is obtained as

$$P_{cr} = \frac{(4.49)^2 EI}{L^2} = \frac{20.16EI}{L^2}$$

From (7.40), (7.38a) and (7.38b) we have

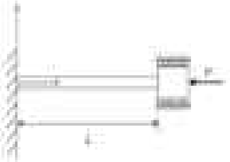
$C_2 = -4.49C_1$

$C_3 = -kC_1 = -\frac{4.49}{L}C_1$

$C_4 = 4.49C_1$

[illegible]

Clamped-Clamped Bar



The boundary (end) conditions for the clamped-clamped bar (Figure 8.7) are $w = 0$, and $dw/dx = 0$ at both ends ($x = 0, L$). Using the general solution (8.16), these boundary conditions become

$$C_2 + C_4 = 0 \qquad \text{[illegible]}$$

$$C_1 k + C_3 = 0 \qquad \text{[illegible]}$$

$$C_1 \sin kL + C_2 \cos kL + C_3 L + C_4 = 0 \qquad \text{[illegible]}$$

$$C_1 k \cos kL - C_2 k \sin kL + C_3 = 0 \qquad \text{[illegible]}$$

Eliminating C1 and C4 from Eqs. (8.45c) and (8.45d) using Eqs. (8.45a) and (8.45b), we obtain

$$C_2(\sin kL - kL) + C_3(\cos kL - 1) = 0 \quad (8.46a)$$

$$C_2(\cos kL - 1) - C_3 \sin kL = 0 \quad (8.46b)$$

Consider the possible solution

$$\sin kL = 0, \quad kL = n\pi, \quad n = 1, 2, 3, \ldots \quad (8.47)$$

For $n = 1, 3, 5, \ldots$, we have

$$\cos kL = -1 \quad (8.48)$$

From Eqs. (8.46a) and (8.46b), the conditions (8.47) and (8.48) require that C1 = 0 and C2 = 0. Thus, we have a trivial solution.

For $n = 2, 4, 6, \ldots$ (or $n = 2s$, $s = 1, 2, 3, \ldots$) we have

$$\sin kL = 0 \quad \text{and} \quad \cos kL = 1 \quad (8.49)$$

Substitution of Eq. (8.49) into Eq. (8.46) yields C2 = 0 (then C1 = 0) and C3 ≠ 0. Thus the buckling mode shape is

$$u(x) = C_3 \cos kx + C_4 = C_3(\cos kx - 1)$$

where
$$k = \frac{2n\pi}{L}, \qquad n = 1, 2, 3, \ldots$$

Thus, the buckling loads are

$$P_{cr}^{(n)} = \frac{4n^2\pi^2 EI}{L^2}, \qquad n = 1, 2, 3, \ldots$$

The lowest buckling load

$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

is four times the buckling load for the pinned–pinned bar.

BAR OF UNSYMMETRIC SECTION

Consider a straight bar of unsymmetric section under compression. The assumed perturbed deflection consists of displacement components $v(x)$ in the y direction and $w(x)$ in the z direction.

Take a free body of a small bar element similar to that of Fig. 3.2. The consideration of balance of forces in the y- and z-directions yields

$$\frac{dV_y}{dx} = 0 \qquad (7.84a)$$

and

$$\frac{dV_z}{dx} = 0 \qquad (7.84b)$$

respectively. The balance of moments yields

$$V_y = -\frac{dM_z}{dx} + P\frac{dv}{dx} \qquad (7.85a)$$

$$V_z = \frac{dM_y}{dx} + P\frac{dw}{dx} \qquad (7.85b)$$

Substituting

$$M_y = -EI_{yz}\frac{d^2v}{dx^2} - EI_y\frac{d^2w}{dx^2}$$

$$M_z = -EI_z\frac{d^2v}{dx^2} - EI_{yz}\frac{d^2w}{dx^2}$$

into (7.85) and then (7.84), we obtain the equilibrium equations as

$$EI_z\frac{d^4v}{dx^4} + EI_{yz}\frac{d^4w}{dx^4} + P\frac{d^2v}{dx^2} = 0 \qquad (7.86a)$$

$$EI_y\frac{d^4w}{dx^4} + EI_{yz}\frac{d^4v}{dx^4} + P\frac{d^2w}{dx^2} = 0 \qquad (7.86b)$$

For illustration, we consider a pinned-pinned bar. It is easy to show that the displacements

$$v = C_1 \sin\frac{\pi x}{L} \qquad (7.87a)$$

$$w = C_2 \sin\frac{\pi x}{L} \qquad (7.87b)$$

satisfy the boundary conditions. Thus, they represent a possible buckling mode. Substitution of (7.87) into (7.86) yields

$$C_1\left(EI_z\frac{\pi^4}{L^4} - P\frac{\pi^2}{L^2}\right) + C_2 EI_{yz}\frac{\pi^4}{L^4} = 0 \qquad (7.88a)$$

$$C_1 EI_{yz}\frac{\pi^4}{L^4} + C_2\left(EI_y\frac{\pi^4}{L^4} - P\frac{\pi^2}{L^2}\right) = 0 \qquad (7.88b)$$

Eliminating C_1 from these equations, we obtain

$$C_2\left[\left(EI_y\frac{\pi^2}{L^2} - P\right) - \frac{EI_{yz}^2}{EI_z(\pi^2/L^2) - P} \times \frac{\pi^4}{L^4}\right] = 0$$

For a nontrivial solution, $C_1 \neq 0$, and we have

$$\left(EI_x\frac{\pi^2}{L^2}-P\right)\left(EI_y\frac{\pi^2}{L^2}-P\right)-E^2I_{xy}^2\frac{\pi^4}{L^4}=0$$

which can be rewritten as

$$P^2-\frac{\pi^2E(I_x+I_y)}{L^2}P+\frac{\pi^4E^2}{L^4}(I_xI_y-I_{xy}^2)=0 \qquad (7.89)$$

Two solutions for P are obtained:

$$P=\frac{\pi^2E}{L^2}\left[\frac{1}{2}(I_x+I_y)\pm\frac{1}{2}\sqrt{(I_x+I_y)^2-4(I_xI_y-I_{xy}^2)}\right]$$

The buckling load is the smaller of the two solutions, i.e.:

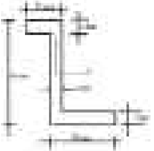
$$P_{cr}=\frac{\pi^2EI_{min}}{L^2} \qquad (7.90)$$

where

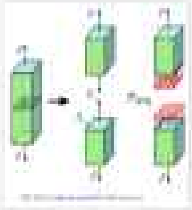
$$I_{min}=\frac{1}{2}(I_x+I_y)-\frac{1}{2}\sqrt{(I_x-I_y)^2+4I_{xy}^2}$$

The quantity I_1 is readily recognized as the smaller moment of inertia about the principal axes of the cross-sectional area.
In fact, I_1 is the minimum value of the moment of inertia of the cross-section about any axis passing through the centroid.
This indicates that buckling deflection takes place in the direction perpendicular to the principal axis about which the moment of inertia is the minimum.

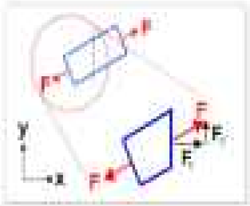
Calculate the buckling load for the given unsymmetrical beam having length of 6 m. take E=70 GPa. Consider both ends hinged boundary condition



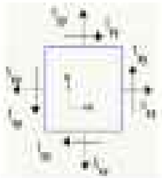
Stress



$$\sigma = \frac{F_{normal}}{A} \quad \text{and} \quad \tau = \frac{F_{parallel}}{A}$$

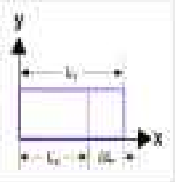


Equilibrium



$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

Strain



$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{x} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_{y} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_{z} \end{bmatrix}$$

Hooke's Law

Hooke's Law describes linear material behavior.

It is commonly used for isotropic materials (same behavior in all directions), but can also be extended to anisotropic materials.

$$\sigma = E\varepsilon$$

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}$$

Add an additional normal stress in the yy direction. Any tensile stress in the yy direction will cause the strain in the xx direction to decrease.

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

Poisson Ratios

Material	Poisson's Ratio	Comments
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
Aluminum	[illegible]	
Copper	[illegible]	[illegible]
Steel	[illegible]	
Cork	[illegible]	[illegible]

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}=\frac{(1-2\nu)}{E}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$$

[illegible]

[illegible]

[illegible]

$$\varepsilon_{vol}=\frac{3(1-2\nu)}{E}\sigma_{hyd}$$

And the ratio of the σ_{hyd} to ε_{vol} is the bulk modulus, K.

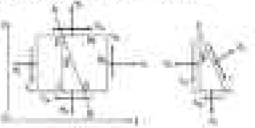
$$K=\frac{\sigma_{hyd}}{\varepsilon_{vol}}=\frac{E}{3(1-2\nu)}$$

[illegible]

$$\frac{E}{K}=3(1-2\nu)$$

Relation between Bulk Modulus and Young's Modulus

DETERMINATION OF STRESSES ON INCLINED PLANES



(a) (b)

[illegible]

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta$$

$$\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta - \tau_{xy} \cos 2\theta$$

Example

[illegible]

Longitudinal stress $(\sigma_L) = \frac{pd}{4t}$ = [illegible]

Circumferential stress $(\sigma_c) = \frac{pd}{2t}$ = [illegible]

The direct stress due to the axial load will contribute to σ_L and is given by

[illegible]

Text Book: Main Resource

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